Geometry of the dynamics of an entangled quantum system

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Sevenans
1. Geometry of quantum dynamics
   - Decomposition of quantum dynamics
   - The base space
   - The principal bundle and its connection
   - Adiabatic quantum dynamics
The Schrödinger equation

\[ i\hbar \dot{\psi} = H(t)\psi \]

\( \psi \in \mathcal{H} \) a Hilbert space (\( \dim \mathcal{H} = n \)); \( H \in \mathcal{L}(\mathcal{H}) \) (self-adjoint).

But:

- \( \|\psi\| = 1 \) (probabilistic interpretation)
- the phase of \( \psi \) has no meaning (only a phase difference).

**Too many unphysical informations into \( \mathcal{H} \) !**
The decomposition

- $P = |\psi\rangle\langle\psi| (||\psi|| = 1)$ rank-1 projection (normalized state without phase).
  \[ i\hbar \dot{P} = [H(t), P] \]

- Cyclic dynamics: $P(T) = P(0)$. Let $\tilde{\psi}(t) \in \mathcal{H}$, be an arbitrary normalized state such that $|\tilde{\psi}\rangle\langle\tilde{\psi}| = P$ and $\tilde{\psi}(T) = \tilde{\psi}(0)$.
  \[ i\hbar \dot{\psi} = H(t)\psi \iff \psi(t) = e^{-i\hbar^{-1}\int_0^{t} \lambda(t')dt'} e^{-\int_0^{t} A(t')dt'} \tilde{\psi}(t) \]
  with $\lambda = \langle\tilde{\psi}|H|\tilde{\psi}\rangle$ and $A = \langle\tilde{\psi}\frac{d}{dt}|\tilde{\psi}\rangle$.
  The phase difference between $\psi(0)$ and $\psi(T)$, $e^{-\int_0^{T} A(t')dt'}$ (geometric phase), is physically meaningful.

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The projective manifold

\[ \psi \in \mathcal{H} \cong \mathbb{C}^n. \| \psi \| = 1 \Rightarrow \psi \in S^{2n-1}. \]
\[ \psi \sim e^{i\varphi} \psi \left( |\psi\rangle \langle \psi | = |e^{i\varphi} \psi\rangle \langle e^{i\varphi} \psi | \right) \Rightarrow P \in S^{2n-1} / \sim \cong \mathbb{C}P^{n-1}. \]

Example: for a 2-level system, \( \mathbb{C}P^1 = S^2 \) (the Bloch sphere):
\[ |\psi\rangle = \cos \theta |0\rangle + e^{i\varphi} \sin \theta |1\rangle, \ (\theta, \varphi) \text{ are local coordinates onto } S^2. \]
Consider all possible arbitrary phases
\[ \{ e^{i\varphi} \psi, |\psi\rangle\langle \psi| = P \in \mathbb{CP}^{n-1} \} \forall \varphi \in [0,2\pi]. \]
⇒ At each point \( P \in \mathbb{CP}^{n-1} \), we attach a copy of \( U(1) \) as manifold (a fibre).

The set of all fibres constitutes the manifold \( S^{2n-1} \) as a fiber space locally diffeomorphic to \( \mathbb{CP}^{n-1} \times U(1) \).

To restore the arbitrary character of the phases, we consider \( U(1) \) (as a group) acting onto \( S^{2n-1} \) as “translations” along the fibres.

The whole structure is a principal \( U(1) \)-bundle \( \mathcal{P} : \)

\[
\begin{array}{ccc}
U(1) & \rightarrow & S^{2n-1} \\
\downarrow \pi & & \downarrow \\
\mathbb{CP}^{n-1} & & \end{array}
\]

with \( \pi(\psi) = |\psi\rangle\langle \psi| \).
The Berry-Simon connection

\[ \tilde{\psi} \text{ s.t. } \pi(\tilde{\psi}) = P \text{ and } \tilde{\psi}(T) = \tilde{\psi}(0), \]
is an arbitrary section of \( P \) above the path \( C \) into \( CP^{n-1} \) (defined by \( \hbar \dot{P} = [H, P] \)).

Let \( A \in \Omega^1 CP^{n-1} \) be defined by \( A(P) = \langle \tilde{\psi} | d | \tilde{\psi} \rangle \) (Berry-Simon gauge potential). The geometric phase is \( e^{-\oint_C A} \) (holonomy of the horizontal lift of \( C \) in the fibres).

\[ e^{-\oint_C A} = e^{-\int_s F} \quad \text{with} \quad \partial S = C, \quad F = \omega A \in \Omega^2 CP^{n-1} \] (Berry-Simon curvature). \((A, F)\) are the local data which define the connective structure of \( P \).

Gauge change : \( \tilde{\psi}' = e^{i\varphi} \tilde{\psi} \Rightarrow A' = A + i \omega \varphi \text{ and } F' = F. \)
Adiabatic quantum dynamics

Adiabatic approximation

\[ i\hbar \dot{\psi} = H(x(t))\psi \]

\( x \in M \) (control manifold).

Adiabatic theorem: \( P(s) = P_\lambda(x(s)) + \mathcal{O}(\frac{1}{T}) \ (s = t/T) \) with

\[ H(x)\phi_\lambda(x) = \lambda(x)\phi_\lambda(x) \ (P_\lambda = |\phi_\lambda\rangle\langle\phi_\lambda|)^2. \]

\[ U(1) \rightarrow \mathcal{P} \]
\[ \downarrow \]
\[ M \]

with gauge potential:

\[ A(x) = \langle \phi_\lambda | \frac{d}{dx^\mu} | \phi_\lambda \rangle dx^\mu \in \Omega^1 M \]

Geometry of quantum dynamics

Dynamics of an entangled quantum system

The categorical bundle

Applications *

Adiabatic quantum dynamics

Interests

- Analogy with classical field theory: $\psi(T) = e^{-\mathcal{L}_A} \psi(0)$ is the transport along $\mathcal{C}$ of a charged particle living in $\mathcal{M}$ where $F = dA$ is a magnetic field. Crossings points $x_* \in \mathcal{M}$ where $\lambda(x_*)$ is locally degenerate with another eigenvalue, appear as magnetic monopoles.

- Adiabatic quantum control: find $\mathcal{C}$ in $\mathcal{M}$ in order to the horizontal lift of $\mathcal{C}$ be the control target.

- Adiabatic quantum computation (AQC): find $\mathcal{C}$ in $\mathcal{M}$ in order to the horizontal lift of $\mathcal{C}$ be a quantum computation (example: quantum annealing).

*This application can be extended to the non-adiabatic case: holonomic quantum computation (HQC).*

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2 Dynamics of an entangled quantum system

- Mixed state
- Phases?
- The base space
Dynamics of an entangled quantum system

Geometry of the dynamics of an entangled quantum system

Mixed state

Density matrix

\[ \dot{\psi} = (H_S(t) \otimes 1_E + 1_S \otimes H_E(t) + V_{int}(t)) \psi \]

with \( \psi \in \mathcal{H}_S \otimes \mathcal{H}_E \), \( H_S \in \mathcal{L}(\mathcal{H}_S) \), \( H_E \in \mathcal{L}(\mathcal{H}_E) \), \( V_{int} \in \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E) \) (\( V_{int} \) modifies the entanglement between \( S \) and \( E \) during the dynamics).

\[ \rho = \text{tr}_E |\psi\rangle \langle \psi| \]

We are interested only by the state of \( S \) (we forget the informations concerning \( E \)).

\( \rho \) is pure (\( \text{tr} \rho^2 = 1 \)) \( \iff \) \( \psi \) is a separable state.

\( \rho \) is mixed (\( \text{tr} \rho^2 < 1 \)) \( \iff \) \( \psi \) is an entangled state.

\( \rho \) is maximally mixed (\( \text{tr} \rho^2 = \frac{1}{n} \)) \( \iff \) \( \psi \) is a “Schrödinger cat”.
\(C^*\)-module structure

**Closed system**

- \(\lambda \psi, \lambda \in \mathbb{C}, \psi \in \mathcal{H}\)
- Field \(\mathbb{C}\)
- Hilbert space \(\mathcal{H}\)
- \(\langle \phi | \psi \rangle \in \mathbb{C}\)
- \(\|\psi\|^2 \in \mathbb{R}^+\)

**Entangled system**

- \(A \otimes 1_E \psi, A \in \mathcal{L}(\mathcal{H}_S), \psi \in \mathcal{H}_S \otimes \mathcal{H}_E\)
- \(C^*\)-algebra \(\mathfrak{a} = \mathcal{L}(\mathcal{H}_S)\)
- \(C^*\)-module \(\mathcal{H}_S \otimes \mathcal{H}_E\)
- \(\langle \phi | \psi \rangle_* = \text{tr}_E |\psi\rangle \langle \phi| \in \mathfrak{a}\)
- \(\|\psi\|^2_* = \rho \in \mathcal{D}(\mathcal{H}_S)\)
Operator valued geometric phases

In the literature:

- **Uhlmann geometric phase** (based on transition probabilities):\(^7\)
  \[
  \tilde{\psi} = 1_S \otimes Pe^{-\int A} \psi \quad \text{with} \quad d\rho = A\rho + \rho A \quad \text{(but a large class of generators are also possible)}\(^8\).
  \[
  \rho = \tilde{\rho}.
  
- **Sjöqvist geometric phase** (based on interferometry):\(^9\)
  \[
  \psi = Pe^{-\int \eta \otimes 1} E \tilde{\psi} \quad \text{with} \quad \eta = \sum_j P_j W^\dagger dW P_j \sigma^{-1} \quad \text{with} \quad \rho = WW^\dagger, \sigma \text{ diagonal matrix of } Sp(\rho) \text{ and } P_j \text{ the associated eigenprojections.}
  \[
  \rho = \tilde{\rho}.
  
- **\(C^*\)-geometric phase** (based on the \(C^*\) inner product):\(^10\)
  \[
  \psi = Pe^{-\int A} \otimes 1 E \tilde{\psi} \quad \text{with} \quad A = \langle \tilde{\psi} | d\tilde{\psi} \rangle_* \| \tilde{\psi} \|^2_*.
  \[
  \rho = Pe^{-\int A} \tilde{\rho} (Pe^{-\int A})^\dagger.
  
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3 notions of phases in the $C^*$-module

$g \in \mathcal{A}$ is a
- phase by invariance if $\|g\psi\|_*^2 = \|\psi\|_*^2$
- phase by equivariance if $\|g\psi\|_*^2 = g\|\psi\|_*^2g^{-1}$
- phase with respect to the Hamiltonian if $Hg\psi = gH\psi$

If $\mathcal{A} = \mathbb{C}$, then the three definitions are the same.

Three definitions of the cyclicity: $\rho(T) = \rho(0)$, $\rho(T) = g\rho(0)g^{-1}$ (isospectral density matrices) or $\rho(T) = g\rho(0)g^\dagger$ (with $[H(0), g] = 0$ and $H(0) = H(T)$ for example).
The categorical bundle

In fact: $\mathcal{D}(\mathbb{C}^n) \rightarrow \Sigma(n)$ is a fibre bundle, but this one is not locally trivial, it is stratified$^{11}$:

$\mathcal{D}(\mathbb{C}^n)$ manifold of density matrices or $\Sigma(n)$ simplex of possible specta?

$^{11}$ I. Bengtsson & K. Zyczkowski, Geometry of quantum states (Cambridge University Press, 2006)
Let $\mathcal{M}$ be the category defined by $\text{Obj} \mathcal{M} = \mathcal{D}(\mathbb{C}^n)$ and $\text{Morph} \mathcal{M} = G \times \mathcal{D}(\mathbb{C}^n)$ (with $G$ the group of phases by equivariance or w.r.t. the Hamiltonian); with the source, the target and the identity maps defined by

\[
\begin{align*}
    s(g, \rho) &= \rho; \\
    t(g, \rho) &= g \rho g^\dagger; \\
    \text{id}_\rho &= (1_S, \rho) \\
    (g', g \rho g^\dagger) \circ (g, \rho) &= (g'g, \rho)
\end{align*}
\]

$g$ is then not viewed as phase (gauge) change but as an arrow of the base category.
3 The categorical bundle

- Fibers
- Connections
- The adiabatic case
Groups acting on the fibres

- $G$ group of phases by equivariance, acting on the left.
- $H$ group of phases by invariance, acting on the left.
- $K$ group of phases by invariance, acting on the right.

$G$ groupoid on the left with $\text{Obj}G = G$, $\text{Morph}G = G \rtimes H$ and

\[
s(g, h) = g; \quad t(g, h) = gh; \quad \text{id}_g = (g, 1_s)
\]

\[
(gh, h') \circ (g, h) = (g, hh')
\]

\[
(g', h')(g, h) = (g'g, g^{-1}h'gh)
\]
The fibred structure

\[ G \Rightarrow P \leftarrow K \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \Sigma \]

\( P \) cannot be defined as a manifold, it is just defined as a category. The structure is a “stratified categorical composite principal bi-bundle”\(^{12}\)

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The gauge potentials

- $\mathcal{A} \in \Omega^1(\Sigma, g)$ ($C^*$-geometric phase generator) : left gauge potential

$$\tilde{\psi}' = g \otimes k \tilde{\psi} \Rightarrow \mathcal{A}' = gA^{-1} + dg^{-1} + g \eta k^{-1} \quad (\eta_k = \langle \tilde{\psi}|k^{-1}dk|\tilde{\psi}\rangle_{\tilde{\rho}}^{-1})$$

- $A \in \Omega^1(\text{Obj} \mathcal{M}, \xi)$ (Uhlmann phase generator) : right object gauge potential

$$\tilde{\psi}' = g \otimes k \tilde{\psi} \Rightarrow A' = k^{-1}A k + k^{-1}dk + k^{-1} \eta g \quad (\eta_g = \langle \tilde{\psi}|dg^{-1}|\tilde{\psi}\rangle_{\tilde{\rho}}^{-1})$$

- $\eta_{\rightarrow} \in \Omega^1(\text{Morph} \mathcal{M}, \xi)$ (Uhlmann generator transformation) : right arrow gauge potential

$$\tilde{\psi}' = h \otimes k \tilde{\psi} \Rightarrow \eta_{\rightarrow}' = (h_{\tilde{\psi}k})^{-1} \eta_{\rightarrow} h_{\tilde{\psi}k} + (h_{\tilde{\psi}k})^{-1} d(h_{\tilde{\psi}k}) + \eta_{h_{\tilde{\psi}k}}$$

$A + \eta_{\rightarrow}$ is the Sjöqvist phase generator (for some $\eta_{\rightarrow}$). The connective structure presents also left/right fake/true curvatures and curvings.
Non-commutative eigenvalues

\[ H(x)\phi_\Lambda(x) = \Lambda(x) \otimes 1_E \phi_\Lambda(x), \quad [\Lambda(x) \otimes 1_E, H(x)] \phi_\Lambda(x) = 0 \]

with \( H(x) = H_S(x) \otimes 1_E + 1_S \otimes H_E(x) + V_{int}(x), \phi_\Lambda \in \mathcal{H}_S \otimes \mathcal{H}_E, \Lambda \in a. \)

\( G \) is the group of phase w.r.t. \( H \) (leaving invariant \( \ker(H - \Lambda \otimes 1_E) \)), \( K \) is the group of unitary operators of \( E \) leaving invariant \( H(x) \).

\[ G \Rightarrow P \leftarrow K \]
\[ \downarrow \]
\[ M \]

\( M \) being the manifold of all configurations of \( x \).

Adiabatic theorem for density matrix

\[ \rho(s) = g_{\Lambda} \varphi_{\Lambda}(s) \rho_{\Lambda}(x(s)) g_{\Lambda} \varphi_{\Lambda}(s)^\dagger + O\left( \max \left( \frac{1}{T}, \epsilon \right) \right) \]

\[ \rho_{\Lambda} = \| \varphi_{\Lambda} \|^2 \text{ and } g_{\Lambda} \varphi_{\Lambda} = T e^{-i \hbar^{-1} T \int_0^s \Lambda(t') dt} P e - \int_C \mathfrak{A}, \]
\[ \mathfrak{A} = \langle \varphi_{\Lambda} | d | \varphi_{\Lambda} \rangle^* \rho_{\Lambda}^{-1} \]

\[ s = t/T \text{ and } \inf_s \min_{b \neq a} | \mu_b - \mu_a | = O(\epsilon) \text{ with } \{ \mu_a \}
\]

4 Applications

- Adiabatic quantum control hampered by entanglement
- Decoherence phenomenons
- Quantum black holes
Interpretation of the connective structure

\[ \mathfrak{A} = \langle \phi_\lambda | d | \phi_\lambda \rangle \ast \rho_\lambda^{-1} \] (left gauge potential).

\[ A = \langle \phi_\lambda | P_\lambda d | \phi_\lambda \rangle \ast \rho_\lambda^{-1} \] (reduced left gauge potential), \( P_\lambda \) is the orthogonal projection onto \( \ker(H - \lambda 1_S \otimes 1_E) \).

- \( B = d\mathfrak{A} - \mathfrak{A} \wedge \mathfrak{A} \) (curving):
  \( \text{tr}_S(\rho_\lambda(x)B(x)) \) measures the entanglement entropy increase induced by variations in the neighbourhood of \( x \).

- \( F = dA - A \wedge A - B \) (fake curvature):
  \( \text{tr}_S(\rho_\lambda(x)F(x)) \) measures the non-adiabaticity in the neighbourhood of \( x \).
Charts of the curving

Average curvings \((\text{tr}(\rho_a B_a))\) for STIRAP control of a 3-level atom entangled with another one\(^{15}\).

\[
S(t) = -\text{tr}(\rho(t) \ln \rho(t))
\]


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True decoherence

True decoherence: irreversible fall of the quantum state purity \( (\text{tr} \rho^2) \) induced by the entanglement of \( S \) with a very large environment \( B \) (thermal bath for example), \( \rho = \text{tr}_B |\psi_{S\otimes B}\rangle\langle\psi_{S\otimes B}|. \)

Master equation:

\[
\dot{\rho} = [H_S, \rho] - \frac{i}{2} \sum_k \gamma^k \{\Gamma_k^\dagger \Gamma_k, \rho\} + i \sum_k \gamma^k \Gamma_k \rho \Gamma_k^\dagger
\]

Schmidt purification procedure: \( \psi \in \mathcal{H}_S \otimes \mathcal{H}_E \) where \( \dim \mathcal{H}_E = \dim \mathcal{H}_S \) (\( E \) is an anchor system) with \( \text{tr}_E |\psi\rangle\langle\psi| = \rho. \)

\[
\dot{\psi} = \left( H_S \otimes 1_E - \frac{i}{2} \sum_k \gamma^k \Gamma_k^\dagger \Gamma_k \otimes 1_E + \frac{i}{2} \sum_k \gamma^k \Gamma_k \otimes \Gamma_k^\dagger (\psi) \right) \psi
\]

where \( \Gamma_k^\dagger \) is a nonlinear operator built with \( \Gamma_k. \)

Fake decoherence

Fake decoherence: irreversible fall of the state purity ($\text{tr} \rho^2$) induced by averaging effect on a lot of copies of $S$,

$$\rho = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} |\psi_{S_n}\rangle\langle\psi_{S_n}|.$$  

The different copies are in different states $\{\psi_{S_n}\}_n$ because of random initial conditions and/or a Hamiltonian $H_S(\varphi^t(x_0))$ which depends on a chaotic or stochastic classical flow $\varphi^t : \Gamma \to \Gamma$ (classical noises).

Schrödinger-Koopman representation: $\psi \in \mathcal{H}_S \otimes L^2(\Gamma, d\mu(x))$.  

$$i\hbar \dot{\psi} = (H_S(x) - i\hbar 1_S \otimes F^\mu(x) \partial_\mu) \psi$$

$$(\dot{\varphi}(x_0) = F(\varphi(x_0))).$$

$$\rho = \text{tr}_{L^2(\Gamma, d\mu(x))} |\psi\rangle \langle \psi|$$

Decoherence phenomena

Quasienergy states

Eigenvector of $H_S(\theta) - i\hbar F^\mu(\theta) \partial_\mu$ with $S$ a spin-$\frac{1}{2}$ kicked by ultra-fast magnetic pulses noised by a chaotic process (defined by the Chirikov-Taylor standard map on the torus).

$\Rightarrow$ definition of ergodic geometric phases in the chaotic sea (usual cyclic geometric phases in the islands of stability).\(^\text{18}\)

Quantum black hole: BFSS matrix model

Matrix model theory (String M-theory in non-perturbative sector):

Event Horizon = thermalized D2-brane, i.e. stack of quantum bosonic strings. Test particle = fermionic string linking the D2-brane to a probe D0-brane.

D2-brane = non-commutative manifold described by \((X^1, X^2, X^3)\) ("space coordinate" operators of the manifold), \(X^i \in \mathcal{L}(\mathbb{C}^N)\) (\(N - 1\) is the number of strings of the stack).

Non-commutative Klein-Gordon and Dirac equations:

\[
\ddot{X}^i - [X^j, [X^i, X_j]] = 0
\]

\[
i|\dot{\psi}\rangle = \sigma_i \otimes (X^i - x^i)|\psi\rangle
\]

\(\psi \in \mathbb{C}^2 \otimes \mathbb{C}^N\) (state of the fermionic string). \(\Rightarrow\) entanglement between the spin of the fermionic string and the D2-brane (event horizon).  

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Quantum black holes

Categorical bundle

\[ \sigma_i \otimes X^i \phi_\Lambda(x) = \Lambda(x) \otimes 1_{\mathbb{C}^N} \phi_\Lambda(x), \quad \Lambda(x) = x^i \sigma_i \]

\[ \mathcal{A} = \langle \phi_\Lambda | d | \phi_\Lambda \rangle * \rho_\Lambda^{-1} \text{ (left gauge potential)} \]

\[ A = \langle \phi_\Lambda | P_\Lambda d | \phi_\Lambda \rangle * \rho_\Lambda^{-1} \text{ (reduced left gauge potential)} \]

The differential manifold \( M_\Lambda \) of all points \( x \) for which \( x^i \sigma_i \) is a non-commutative eigenvalue is the emergent “classical” event horizon.

Let \( f = d \text{tr}_{\mathbb{C}^2}(\rho_\Lambda \mathcal{A}) \), and \( \theta = f^{-1} \left( f_{ij} \theta^{jk} = \delta^k_i \right) \); then

\[ dt^2 - \theta^{ik} \theta^{jl} g_{kl} dx_i dx_j \]

is the space-time metric (\( g_{kl} \) is the metric of \( M_\Lambda \) induced by its embedding into \( \mathbb{R}^3 \)).

The curving \( B = d\mathcal{A} - \mathcal{A} \wedge \mathcal{A} - F \) (\( F = dA - A \wedge A \)), is the Kalb-Ramon (Neveu-Schwarz-Neveu-Schwarz) field (torsion potential / axion field).

Dilaton field in the gauge structure???

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20. work in progress...

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5 Conclusion
Conclusion

Forthcoming works:

- **Decoherence**: make more easy the computation of the quasienergy states.
- **Quantum black hole**: finish the study of the geometric structure.
- **General theory**: generalize AQC with entangled (or noised) quantum systems.
- **Open question**: relation between categorical geometry, non-commutative geometry and algebraic geometry (works of Frédéric).