Geometry of the dynamics of an entangled quantum system

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The Schrödinger equation

$$\imath\hbar\dot{\psi} = H(t)\psi$$

$\psi \in \mathcal{H}$ a Hilbert space $(\dim \mathcal{H} = n)$; $H \in \mathcal{L}(H)$ (self-adjoint).
But :

- $\|\psi\| = 1$ (probabilistic interpretation)
- the phase of ψ has no meaning (only a phase difference).

Too many unphysical informations into $\mathcal{H}$!

The decomposition

- $P = |\psi\rangle\langle\psi|$ ($\|\psi\| = 1$) rank-1 projection (normalized state without phase).

$$\imath\hbar\dot{P} = [H(t), P]$$

- Cyclic dynamics : $P(T) = P(0)$. Let $\tilde{\psi}(t) \in \mathcal{H}$, be an arbitratry normalized state such that $|\tilde{\psi}\rangle\langle\tilde{\psi}| = P$ and $\tilde{\psi}(T) = \tilde{\psi}(0)$.

$$\imath\hbar\dot{\psi} = H(t)\psi \iff \psi(t) = e^{-\imath\hbar^{-1}\int_0^t \lambda(t')dt'} e^{-\int_0^t A(t')dt'} \tilde{\psi}(t)$$

with $\lambda = \langle\tilde{\psi}|H|\tilde{\psi}\rangle$ and $A = \langle\tilde{\psi}|\frac{d}{dt}|\tilde{\psi}\rangle$.
The phase difference between $\psi(0)$ and $\psi(T)$, $e^{-\int_0^T A(t')dt'}$ (geometric phase), is physically meaningful [1].

1. Y. Aharonov & J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987)

The projective manifold

$\psi \in \mathcal{H} \simeq \mathbb{C}^n$. $\|\psi\| = 1 \Rightarrow \psi \in S^{2n-1}$.
$\psi \sim e^{\imath\varphi}\psi$ $(|\psi\rangle\langle\psi| = |e^{\imath\varphi}\psi\rangle\langle e^{\imath\varphi}\psi|) \Rightarrow P \in S^{2n-1}/\sim \simeq \mathbb{C}P^{n-1}$.

Example : for a 2-level system, $\mathbb{C}P^1 = S^2$ (the Bloch sphere) :
$|\psi\rangle = \cos\theta|0\rangle + e^{\imath\varphi}\sin\theta|1\rangle$, (θ, φ) are local coordinates onto S^2.

$U(1)$-bundle

Consider all possible arbitrary phases
$\{e^{\imath\varphi}\psi, |\psi\rangle\langle\psi| = P \in \mathbb{C}P^{n-1}\}_{\varphi\in[0,2\pi[}$.
$\Rightarrow$ At each point $P \in \mathbb{C}P^{n-1}$, we attach a copy of $U(1)$ as manifold (a fibre).
The set of all fibres constitutes the manifold S^{2n-1} as a fiber space locally diffeomorph to $\mathbb{C}P^{n-1} \times U(1)$.
To restore the arbitrary character of the phases, we consider $U(1)$ (as a group) acting onto S^{2n-1} as “translations” along the fibres.
The whole structure is a principal $U(1)$-bundle $\mathcal{P}$:

$$\begin{array}{ccc} U(1) & \rightarrow & S^{2n-1} \\ & & \downarrow \pi \\ & & \mathbb{C}P^{n-1} \end{array}$$

with $\pi(\psi) = |\psi\rangle\langle\psi|$.

The Berry-Simon connection

$\tilde{\psi}$ s.t. $\pi(\tilde{\psi}) = P$ and $\tilde{\psi}(T) = \tilde{\psi}(0)$, is an arbitrary section of $\mathcal{P}$ above the path $\mathcal{C}$ into $\mathbb{C}P^{n-1}$ (defined by $\imath\hbar\dot{P} = [H, P]$).

Let $A \in \Omega^1\mathbb{C}P^{n-1}$ be defined by $A(P) = \langle\tilde{\psi}|d|\tilde{\psi}\rangle$ (Berry-Simon gauge potential). The geometric phase is $e^{-\oint_{\mathcal{C}} A}$ (holonomy of the horizontal lift of $\mathcal{C}$ in the fibres).

$e^{-\oint_{\mathcal{C}} A} = e^{-\iint_{\mathcal{S}} F}$ with $\partial\mathcal{S} = \mathcal{C}$, $F = dA \in \Omega^2\mathbb{C}P^{n-1}$ (Berry-Simon curvature). (A, F) are the local data which define the connective structure of $\mathcal{P}$.

Gauge change : $\tilde{\psi}' = e^{\imath\varphi}\tilde{\psi} \Rightarrow A' = A + \imath d\varphi$ and $F' = F$.

Adiabatic approximation

$$\imath\hbar\dot{\psi} = H(x(t))\psi$$

$x \in M$ (control manifold).
Adiabatic theorem : $P(s) = P_\lambda(x(s)) + \mathcal{O}(\frac{1}{T})$ $(s = t/T)$ with $H(x)\phi_\lambda(x) = \lambda(x)\phi_\lambda(x)$ $(P_\lambda = |\phi_\lambda\rangle\langle\phi_\lambda|)$ [2].

$$\begin{array}{ccc} U(1) & \to & \mathcal{P} \\ & & \downarrow \\ & & M \end{array}$$

with gauge potential :

$$A(x) = \langle\phi_\lambda|\frac{d}{dx^\mu}|\phi_\lambda\rangle dx^\mu \in \Omega^1 M$$

2. B. Simon, Phys. Rev. Lett. **51**, 2167 (1983)

Interests

- Analogy with classical field theory : $\psi(T) = e^{-\oint_{\mathcal{C}} A}\psi(0)$ is the transport along $\mathcal{C}$ of a charged particle living in $\mathcal{M}$ where $F = dA$ is a magnetic field. Crossings points $x_* \in \mathcal{M}$ where $\lambda(x_*)$ is locally degenerate with another eigenvalue, appear as magnetic monopoles [3].
- Adiabatic quantum control : find $\mathcal{C}$ in $\mathcal{M}$ in order to the horizontal lift of $\mathcal{C}$ be the control target [4].
- Adiabatic quantum computation (AQC) : find $\mathcal{C}$ in $\mathcal{M}$ in order to the horizontal lift of $\mathcal{C}$ be a quantum computation [5] (example : quantum annealing [6]).
  This application can be extended to the non-adiabatic case : holonomic quantum computation (HQC).

3. F. Wilczeck & A. Zee, Phys. Rev. Lett. **52**, 2111 (1984)
4. U. Boscain *etal*, arXiv :1102.3063 (2011)
5. E. Farhi *etal*, arXiv :quant-ph/0001106 (2000)
6. S. Morita & H. Nishimori, arXiv :0806.1859 (2008)

2 Dynamics of an entangled quantum system

- Mixed state
- Phases ?
- The base space

Density matrix

$$\imath\hbar\dot{\psi} = (H_S(t) \otimes 1_E + 1_S \otimes H_E(t) + V_{int}(t))\,\psi$$

with $\psi \in \mathcal{H}_S \otimes H_E$, $H_S \in \mathcal{L}(\mathcal{H}_S)$, $H_E \in \mathcal{L}(\mathcal{H}_E)$, $V_{int} \in \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$ (V_{int} modifies the entanglement between S and E during the dynamics).

$$\rho = \mathrm{tr}_E |\psi\rangle\!\rangle\langle\!\langle\psi|$$

We are interested only by the state of S (we forget the informations concerning E).
ρ is pure ($\mathrm{tr}\rho^2 = 1$) $\iff$ ψ is a separable state.
ρ is mixed ($\mathrm{tr}\rho^2 < 1$) $\iff$ ψ is an entangled state.
ρ is maximaly mixed ($\mathrm{tr}\rho^2 = \frac{1}{n}$) $\iff$ ψ is a "Schrödinger cat".

C^*-module structure

Closed system	**Entangled system**
$\lambda\psi$, $\lambda \in \mathbb{C}$, $\psi \in \mathcal{H}$	$A \otimes 1_E \psi$, $A \in \mathcal{L}(\mathcal{H}_S)$, $\psi \in \mathcal{H}_S \otimes \mathcal{H}_E$
field $\mathbb{C}$	C^*-algebra $\mathfrak{a} = \mathcal{L}(\mathcal{H}_S)$
Hilbert space $\mathcal{H}$	C^*-module $\mathcal{H}_S \otimes \mathcal{H}_E$
$\langle\phi\|\psi\rangle \in \mathbb{C}$	$\langle\phi\|\psi\rangle_* = \mathrm{tr}_E\|\psi\rangle\rangle\langle\langle\phi\| \in \mathfrak{a}$
$\|\psi\|^2 \in \mathbb{R}^+$	$\|\psi\|_*^2 = \rho \in \mathcal{D}(\mathcal{H}_S)$

Operator valued geometric phases

In the litterature :

- Uhlmann geometric phase (based on transition probabilities) [7] :
  $\tilde\psi = 1_S \otimes \mathbb{P}e^{-\int A}\psi$ with $d\rho = A\rho + \rho A$ (but a large class of generators are also possible [8]).
  $\rho = \tilde\rho$
- Sjöqvist geometric phase (based on interferometry) [9] :
  $\psi = \mathbb{P}e^{-\int \eta} \otimes 1_E \tilde\psi$ with $\eta = \sum_j P_j W^\dagger dW P_j \sigma^{-1}$ with $\rho = WW^\dagger$, σ diagonal matrix of $\mathrm{Sp}(\rho)$ and P_j the associated eigenprojections.
  $\rho = \tilde\rho$.
- C^*-geometric phase (based on the C^* inner product) [10] :
  $\psi = \mathbb{P}e^{-\int \mathfrak{A}} \otimes 1_E \tilde\psi$ with $\mathfrak{A} = \langle\tilde\psi|d\tilde\psi\rangle_* \|\tilde\psi\|_*^{-2}$.
  $\rho = \mathbb{P}e^{-\int \mathfrak{A}} \tilde\rho (\mathbb{P}e^{-\int \mathfrak{A}})^\dagger$.

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7. A. Uhlmann, Rep. Math. Phys. **24**, 229 (1986)
8. J. Dittman, G. Rudolph, J. Math. Phys. **33**, 4148 (1992)
9. E. Sjöqvist *etal*, Phys. Rev. Lett. **85**, 2845 (2000)
10. D. Viennot & J. Lages, J. Phys. A **44**, 365301 (2011)

3 notions of phases in the C^*-module

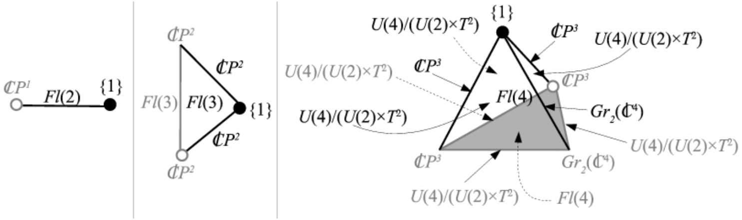
$g \in \mathfrak{a}$ is a

- phase by invariance if $\|g\psi\|_*^2 = \|\psi\|_*^2$
- phase by equivariance if $\|g\psi\|_*^2 = g\|\psi\|_*^2 g^{-1}$
- phase with respect to the Hamiltonian if $Hg\psi = gH\psi$

If $\mathfrak{a} = \mathbb{C}$, then the three definitions are the same.
Three definitions of the cyclicity : $\rho(T) = \rho(0)$, $\rho(T) = g\rho(0)g^{-1}$ (isospectral density matrices) or $\rho(T) = g\rho(0)g^\dagger$ (with $[H(0), g] = 0$ and $H(0) = H(T)$ for example).

The base space

$\mathcal{D}(\mathbb{C}^n)$ manifold of density matrices or $\Sigma(n)$ simplex of possible specta ?
In fact : $\mathcal{D}(\mathbb{C}^n) \to \Sigma(n)$ is a fibre bundle, but this one is not locally trivial, it is stratified [11] :



11. I. Bengtsson & K. Zyczkowski, Geometry of quantum states (Cambridge University Press, 2006)

The base space as a category

Let $\mathscr{M}$ be the category defined by $\mathrm{Obj}\mathscr{M} = \mathcal{D}(\mathbb{C}^n)$ and $\mathrm{Morph}\mathscr{M} = G \times \mathcal{D}(\mathbb{C}^n)$ (with G the group of phases by equivariance or w.r.t. the Hamiltonian) ; with the source, the target and the identity maps defined by

$$s(g,\rho) = \rho; \quad t(g,\rho) = g\rho g^\dagger, \quad \mathrm{id}_\rho = (1_S, \rho)$$

$$(g', g\rho g^\dagger) \circ (g,\rho) = (g'g, \rho)$$

g is then not viewed as phase (gauge) change but as an arrow of the base category.

3 The categorical bundle

- Fibers
- Connections
- The adiabatic case

Groups acting on the fibres

- G group of phases by equivariance, acting on the left.
- H group of phases by invariance, acting on the left.
- K group of phases by invariance, acting on the right.

$\mathscr{G}$ groupoid on the left with $\mathrm{Obj}\mathscr{G} = G$, $\mathrm{Morph}\mathscr{G} = G \rtimes H$ and

$$s(g,h) = g; \quad t(g,h) = gh; \quad \mathrm{id}_g = (g, 1_S)$$

$$(gh, h') \circ (g,h) = (g, hh')$$

$$(g', h')(g,h) = (g'g, g^{-1}h'gh)$$

The fibred structure

$$\begin{array}{ccccc} \mathscr{G} & \Rightarrow & \mathscr{P} & \leftarrow & K \\ & & \downarrow \quad \Downarrow & & \\ & & \downarrow \ \mathscr{M} & & \\ & & \downarrow \quad \downarrow & & \\ & & \Sigma & & \end{array}$$

$\mathscr{P}$ cannot be defined as a manifold, it is just defined as a category.
The structure is a "stratified categorical composite principal bi-bundle"[12]

12. D. Viennot, J. Geom. Phys. **133**, 42 (2018)

The gauge potentials

- $\mathfrak{A} \in \Omega^1(\Sigma, \mathfrak{g})$ (C^*-geometric phase generator) : left gauge potential

$$\tilde{\psi}' = g \otimes k \tilde{\psi} \Rightarrow \mathfrak{A}' = g\mathfrak{A}g^{-1} + dgg^{-1} + g\eta_k g^{-1} \quad (\eta_k = \langle \tilde{\psi} | k^{-1} dk | \tilde{\psi} \rangle_* \tilde{\rho}^{-1})$$

- $A \in \Omega^1(\mathrm{Obj}\mathscr{M}, \mathfrak{k})$ (Uhlmann phase generator) : right object gauge potential

$$\tilde{\psi}' = g \otimes k \tilde{\psi} \Rightarrow \mathfrak{A}' = k^{-1}\mathfrak{A}k + k^{-1}dk + k^{-1}\eta_g k \quad (\eta_g = \langle \tilde{\psi} | dgg^{-1} | \tilde{\psi} \rangle_* \tilde{\rho}^{-1})$$

- $\eta_\rightarrow \in \Omega^1(\mathrm{Morph}\mathscr{M}, \mathfrak{k})$ (Uhlmann generator transformation) : right arrow gauge potential

$$\tilde{\psi}' = h \otimes k \tilde{\psi} \Rightarrow \eta'_\rightarrow = (h_{\tilde{\psi}} k)^{-1} \eta_\rightarrow h_{\tilde{\psi}} k + (h_{\tilde{\psi}} k)^{-1} d(h_{\tilde{\psi}} k) + \eta_{h_{\tilde{\psi}} k}$$

$A + \eta_\rightarrow$ is the Sjöqvist phase generator (for some $\eta_\rightarrow$). The connective structure presents also left/right fake/true curvatures and curvings.

Non-commutative eigenvalues

$$H(x)\phi_\Lambda(x) = \Lambda(x) \otimes 1_E \phi_\Lambda(x), \qquad [\Lambda(x) \otimes 1_E, H(x)]\phi_\Lambda(x) = 0$$

with $H(x) = H_S(x) \otimes 1_E + 1_S \otimes H_E(x) + V_{int}(x)$, $\phi_\Lambda \in \mathcal{H}_S \otimes \mathcal{H}_E$, $\Lambda \in \mathfrak{a}$.
G is the group of phase w.r.t. H (leaving invariant $\ker(H - \Lambda \otimes 1_E)$), K is the group of unitary operators of E leaving invariant $H(x)$.

$$\begin{array}{ccccc} \mathscr{G} & \Rightarrow & \mathscr{P} & \leftarrow & K \\ & & \downarrow & & \\ & & M & & \end{array}$$

M being the manifold of all configurations of x [13].

13. D. Viennot & J. Lages, J. Phys. A **44**, 365301 (2011)

Adiabatic theorem for density matrix

$$\rho(s) = g_{\Lambda\mathfrak{A}}(s)\rho_\Lambda(x(s))g_{\Lambda\mathfrak{A}}(s)^\dagger + \mathcal{O}\left(\max\left(\frac{1}{T}, \epsilon\right)\right)$$

$\rho_\Lambda = \|\phi_\Lambda\|_*^2$ and $g_{\Lambda\mathfrak{A}} = \mathbb{T}e^{-\imath\hbar^{-1}T\int_0^s \Lambda(t')dt}\mathbb{P}e^{-\int_{\mathcal{C}}\mathfrak{A}}$,
$\mathfrak{A} = \langle\phi_\Lambda|d|\phi_\Lambda\rangle_*\rho_\Lambda^{-1}$. [14]
$s = t/T$ and $\inf_s \min_{b\neq a}|\mu_b - \mu_a| = \mathcal{O}(\epsilon)$ with $\{\mu_a\}_a = \mathrm{Sp}(H_S)$.

14. D. Viennot & L. Aubourg, J. Phys. A **48**, 025301 (2015)

4 Applications

- Adiabatic quantum control hampered by entanglement
- Decoherence phenomenons
- Quantum black holes

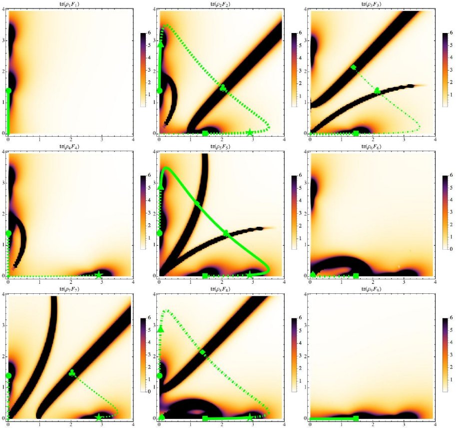
Interpretation of the connective structure

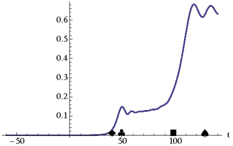
$\mathfrak{A} = \langle \phi_\lambda | d | \phi_\lambda \rangle_* \rho_\lambda^{-1}$ (left gauge potential).
$A = \langle \phi_\lambda | P_\lambda d | \phi_\lambda \rangle_* \rho_\lambda^{-1}$ (reduced left gauge potential), P_λ is the orthogonal projection onto $\ker(H - \lambda 1_S \otimes 1_E)$.

- $B = d\mathfrak{A} - \mathfrak{A} \wedge \mathfrak{A}$ (curving) :
  $\mathrm{tr}_S\left(\rho_\lambda(x) B(x)\right)$ measures of the entanglement entropy increase induced by variations in the neighbourhood of x.
- $F = dA - A \wedge A - B$ (fake curvature) :
  $\mathrm{tr}_S\left(\rho_\lambda(x) F(x)\right)$ measures the non-adiabaticy in the neighbourhood of x.

Charts of the curving

Average curvings ($\mathrm{tr}(\rho_a B_a)$) for STIRAP control of a 3-level atom entangled with another one [15].





$$S(t) = -\mathrm{tr}(\rho(t) \ln \rho(t))$$

15. D. Viennot, J. Phys. A **47**, 295301 (2014)

True decoherence

True decoherence : irreversible fall of the quantum state purity $(\mathrm{tr}\rho^2)$ induced by the entanglement of S with a very large environment B (thermal bath for example), $\rho = \mathrm{tr}_B|\psi_{S\otimes B}\rangle\!\rangle\langle\!\langle\psi_{S\otimes B}|$.
Master equation :

$$\imath\hbar\dot\rho = [H_S,\rho] - \frac{\imath}{2}\sum_k \gamma^k\{\Gamma_k^\dagger\Gamma_k,\rho\} + \imath\sum_k \gamma^k\Gamma_k\rho\Gamma_k^\dagger$$

Schmidt purification procedure : $\psi \in \mathcal{H}_S\otimes\mathcal{H}_E$ where $\dim\mathcal{H}_E = \dim\mathcal{H}_S$ (E is an anchor system) with $\mathrm{tr}_E|\psi\rangle\!\rangle\langle\!\langle\psi| = \rho$.

$$\imath\hbar\dot\psi = \left(H_S\otimes 1_E - \frac{\imath}{2}\sum_k \gamma^k\Gamma_k^\dagger\Gamma_k\otimes 1_E + \frac{\imath}{2}\sum_k\gamma^k\Gamma_k\otimes\Gamma_k^\ddagger(\psi)\right)\psi$$

where $\Gamma_k^\ddagger$ is a nonlinear operator built with $\Gamma_k^\dagger$. [16]

16. D. Viennot, J. Geom. Phys. **133**, 42 (2018)

Fake decoherence

Fake decoherence : irreversible fall of the state purity $(\mathrm{tr}\rho^2)$ induced by averaging effect on a lot of copies of S,
$\rho = \lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^N |\psi_{S_n}\rangle\langle\psi_{S_n}|$.
The different copies are in different states $\{\psi_{S_n}\}_n$ because of random initial conditions and/or a Hamiltonian $H_S(\varphi^t(x_0))$ which depends on a chaotic or stochastic classical flow $\varphi^t : \Gamma \to \Gamma$ (classical noises).
Schrödinger-Koopman representation : $\psi \in \mathcal{H}_S \otimes L^2(\Gamma, d\mu(x))$. [17]

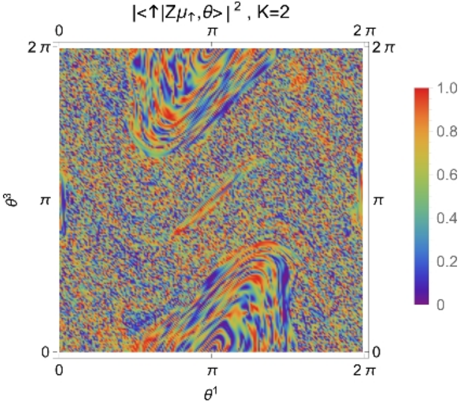
$$\imath\hbar\dot{\psi} = (H_S(x) - \imath\hbar 1_S \otimes F^\mu(x)\partial_\mu)\,\psi$$

$(\dot{\varphi}(x_0) = F(\varphi(x_0)))$.

$$\rho = \mathrm{tr}_{L^2(\Gamma,d\mu(x))}|\psi\rangle\!\rangle\langle\!\langle\psi|$$

17. D. Viennot & L. Aubourg, J. Phys. A **51**, 335201 (2018)

Quasienergy states



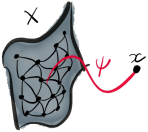
Eigenvector of $H_S(\theta) - \imath\hbar F^\mu(\theta)\partial_\mu$ with S a spin-$\frac{1}{2}$ kicked by ultra-fast magnetic pulses noised by a chaotic process (defined by the Chirikov-Taylor standard map on the torus).

⇒ definition of ergodic geometric phases in the chaotic sea (usual cyclic geometric phases in the islands of stability).[18]

18. D. Viennot & L. Aubourg, J. Phys. A **51**, 335201 (2018)

Quantum black hole : BFSS matrix model

Matrix model theory (String M-theory in non-perturbative sector) :



Event Horizon = thermalized D2-brane, i.e. stack of quantum bosonic strings. Test particle = fermionic string linking the D2-brane to a probe D0-brane

D2-brane = non-commutative manifold described by (X^1, X^2, X^3) ("space coordinate" operators of the manifold), $X^i \in \mathcal{L}(\mathbb{C}^N)$ ($N-1$ is the number of strings of the stack).
Non-commutative Klein-Gordon and Dirac equations :

$$\ddot{X}^i - [X^j, [X^i, X_j]] = 0$$

$$\imath|\dot{\psi}\rangle\rangle = \sigma_i \otimes (X^i - x^i)|\psi\rangle\rangle$$

$\psi \in \mathbb{C}^2 \otimes \mathbb{C}^N$ (state of the fermionic string). $\Rightarrow$ entanglement between the spin of the fermionic string and the D2-brane (event horizon). [19]

19. D. Viennot & L. Aubourg, Class. Quant. Grav. **35**, 135007 (2018)

Categorical bundle

$$\sigma_i \otimes X^i \phi_\Lambda(x) = \Lambda(x) \otimes 1_{\mathbb{C}^N} \phi_\Lambda(x), \quad \Lambda(x) = x^i \sigma_i$$

$\mathfrak{A} = \langle \phi_\Lambda | d | \phi_\Lambda \rangle_* \rho_\Lambda^{-1}$ (left gauge potential).
$A = \langle \phi_\Lambda | P_\Lambda d | \phi_\Lambda \rangle_* \rho_\Lambda^{-1}$ (reduced left gauge potential).
The differential manifold M_Λ of all points x for which $x^i \sigma_i$ is a non-commutative eigenvalue is the emergent "classical" event horizon.
Let $f = d\mathrm{tr}_{\mathbb{C}^2}(\rho_\Lambda \mathfrak{A})$, and $\theta = f^{-1}$ $(f_{ij}\theta^{jk} = \delta_i^k)$; then $dt^2 - \theta^{ik}\theta^{jl} g_{kl} dx_i dx_j$ is the space-time metric (g_{kl} is the metric of M_Λ induced by its embedding into $\mathbb{R}^3$).
The curving $B = d\mathfrak{A} - \mathfrak{A} \wedge \mathfrak{A} - F$ ($\mathcal{F} = dA - A \wedge A$), is the Kalb-Ramon (Neveu-Schwarz-Neveu-Schwarz) field (torsion potential / axion field).
Dilaton field in the gauge structure ? ? ? [20]

20. work in progress...

5 Conclusion

Conclusion

Forthcoming works :

- Decoherence : make more easy the computation of the quasienergy states.
- Quantum black hole : finish the study of the geometric structure.
- General theory : generalize AQC with entangled (or noised) quantum systems.
- Open question : relation between categorical geometry, non-commutative geometry and algebraic geometry (works of Frédéric).