The Koopman approach for dynamical systems and networks

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Besançon

1. The principle of the Koopman approach
   - Context
   - The idea of the Koopman approach
   - First properties of the Koopman operator

Dynamical systems

Definition (Dynamical system)

We call dynamical system the three kinds of data (Γ, φ^t, μ) *where*

- Γ *is a phase space (topological space), supposed* ***compact*** *in this presentation ;*
- $\mathbb{R}^+ \ni t \mapsto \varphi^t \in \mathrm{Aut}\Gamma$ *is the flow of the system (one parameter continuous semigroup of automorphisms of* Γ*) ;*
- $\mu : \mathscr{T} \to [0,1]$ *is a (probability) measure on* Γ *(*$\mathscr{T}$ *is the Borel* σ*-algebra of* Γ*).*

All presented elements can be rewritten for discrete time flows. We denote by $F \in \mathrm{Aut}\Gamma$ the flow generator :

$$\dot{x}(t) = F(x(t)) \iff x(t) = \varphi^t(x(0))$$

In general μ is the micro-canonical distribution (uniform probability distribution on Γ). The flow is not necessary measure preserving.

A well kown idea in quantum mechanics...

- Schrödinger representation
  $\mathcal{H}$: state Hilbert space.
  $\forall \psi_0 \in \mathcal{H}$, $\psi(t) = U(t,0)\psi_0$ ($U(t,0) \in \mathcal{U}(\mathcal{H})$).
- Heisenberg representation
  $\mathfrak{a}$: observable (von Neumann) algebra.
  $\forall A_0 \in \mathfrak{a}$, $A(t) = \mathcal{T}^t(A_0) = U(t,0)A_0U(t,0)^\dagger$.

  States of $\mathfrak{a}$: $\mathcal{E} = \{\omega \in \mathcal{L}(\mathfrak{a},\mathbb{R}), \omega(1) = 1, \omega(A^\dagger A) \geq 0\}$

$$\forall \omega \in E, \exists \rho, \quad \omega(A) = \mathrm{tr}(\rho A)$$

The Koopman approach in classical mechanics

- Usual representation
  Γ : phase space.
  $\forall x_0 \in \Gamma$, $x(t) = \varphi^t(x_0)$ ($\varphi^t \in \mathrm{Aut}\Gamma$).
- Koopman representation
  $\mathfrak{a} = L^\infty(\Gamma, d\mu)$: observable (von Neumann) algebra.
  $\forall f \in \mathfrak{a}$, $f(\varphi^t(x)) = \mathcal{T}^t f(x)$ ($\mathcal{T}^t \in \mathcal{L}(\mathfrak{a})$, Koopman operator).

  States of $\mathfrak{a}$: $\mathcal{E} = \{\mathcal{L}(\mathfrak{a}, \mathbb{R}), \omega(1) = 1, \omega(|f|^2) \geq 0\}$

$$\forall \omega \in \mathcal{E}, \exists \rho \in L^1_+(\Gamma, d\mu), \quad \omega(f) = \langle \rho | f \rangle = \int_\Gamma \rho(x) f(x) d\mu(x)$$

Interest of the Koopman approach

φ^t is a nonlinear automorphism.
$\mathcal{T}^t$ is a linear map.

Price to pay : Γ is a finite dimensional topological space, but $L^\infty(\Gamma, d\mu) = \{f : \Gamma \to \mathbb{C}, \sup_{x\in\Gamma} |f(x)| < \infty\}$ is an infinite dimensional Banach space.

The Perron-Frobenius operator

$\forall f \in L^\infty(\Gamma, d\mu), \forall \rho \in L^1(\Gamma, d\mu)$

$$\langle \rho | \mathcal{T}^t f \rangle = \langle \mathcal{P}^t \rho | f \rangle$$

The $L^1 - L^\infty$-adjoint of the Koopman operator, $\mathcal{P}^t \in \mathcal{L}(L^1(\Gamma, d\mu))$, is called the Perron-Frobenius operator.

Generalisation

Extension of the observable algebra : $\mathfrak{a} = L^\infty(\Gamma, d\mu) \subset L^2(\Gamma, d\mu)$
Restriction of the state space : $\mathcal{E} \simeq L^1(\Gamma, d\mu) \supset L^2(\Gamma, d\mu)$

Interest : $\mathcal{K} = L^2(\Gamma, d\mu)$ is an Hilbert space.

Koopman generator

$$\dot{x}(t) = F(x(t)) \iff \mathcal{T}^t = e^{tF^\mu \partial_\mu}$$

$F^\mu \frac{\partial}{\partial x^\mu} \in \mathcal{L}(\mathcal{K})$ is the generator of the Koopman operator.

If the system is Hamiltonian then $F^\mu \frac{\partial}{\partial x^\mu} = \{\cdot, \mathscr{H}\}$, where $\{\cdot,\cdot\}$ is the Poisson bracket of Γ and $\mathscr{H}$ is the system Hamiltonian.

The ergodic hierarchy

Ergodic hierarchy : "disorder" and "impredictibility" hierarchy of bounded non-cyclic dynamical systems.

ergodic	$\prec$	mixing	$\prec$	exact	$\prec$	Bernoulli
(dense orbit, quasiperiodic)		(chaotic, pseudo-random)		(pseudo-random, irreversible)		(random, stochastic)

Definition

A dynamical system is said :

- *ergodic if* $\forall A \in \mathscr{T}$, $\forall t > 0$, $\varphi^t(A) = A \Rightarrow \mu(A) = 0$ *or* 1.
- *mixing if* $\forall A, B \in \mathscr{T}$, $\lim_{t \to +\infty} \mu(\varphi^t(A) \cap B) = \mu(A)\mu(B)$.
- *exact if* $\forall A \in \mathscr{T}$ $(\mu(A) \neq 0)$, $\lim_{t \to +\infty} \mu(\varphi^t(A)) = 1$.
- *Bernoulli if* $\forall t > 0$, $\forall A, B \in X$ *for a certain partition* X *of* Γ *with elements in* $\mathscr{T}$, $\mu(\varphi^t(A) \cap B) = \mu(A)\mu(B)$.

Unpractical topological definitions !

Definitions in the Koopman approach

A dynamical system is :

- ergodic iff $\forall f \in \mathcal{K}$, for μ-almost all $x_0 \in \Gamma$, we have

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \mathcal{T}^t f(x_0) dt = \langle 1 | f \rangle$$

- mixing iff $\forall f, g \in \mathcal{K}$, we have

$$\lim_{t \to +\infty} \langle g | \mathcal{T}^t f \rangle = \langle g | 1 \rangle \langle 1 | f \rangle$$

- exact iff $\forall \rho \in L^1_+(\Gamma, d\mu)$, we have

$$\lim_{t \to +\infty} \| \mathcal{P}^t \rho - \langle \rho | 1 \rangle \|_1 = 0$$

- Bernoulli iff $\forall \rho \in L^1_+(\Gamma, d\mu)$, we have

$$\langle \mathcal{P}^t \rho | 1 \rangle = \langle \rho | 1 \rangle$$

Measure of disorder

For φ^t preserving the measure μ, the disorder created by time unit by the Koopman operator action is measured by the Maličky-Riečan entropy :

$$h(\mathcal{T}) = \sup_G \limsup_{n \to +\infty} \frac{1}{n} \sum_{g \in G_n} \left(-\langle 1|g\rangle \ln\langle 1|g\rangle + \langle 1|g \ln g\rangle\right)$$

with $G = \{g_i \in L^\infty(\Gamma, d\mu)\}_{i=1,\dots,m}$ a finite partition of the unity, and $G_n = \bigvee_{p=0}^{n-1} \mathcal{T}^{n\tau}(G)$ ($\mathcal{T}^t(G) = \{\mathcal{T}^t g_i\}_i$, $G \vee H = \{g_i h_j\}_{i,j}$).

2 Spectral analysis of the Koopman operator

- The Koopman modes
- The Koopman spectrum

Koopman values and modes

Koopman value $\lambda \in \mathrm{Sp}(F^\mu \partial_\mu)$ associated with the Koopman mode $f_\lambda \in \mathcal{C}^1(\Gamma)$:

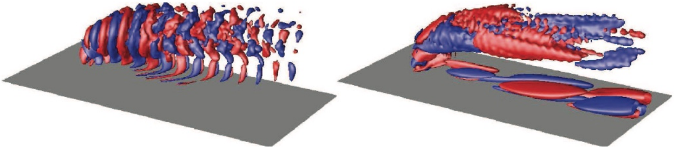
$$F^\mu(x) \frac{\partial f_\lambda(x)}{\partial x^\mu} = \lambda f_\lambda(x)$$

$$\mathcal{T}^t f_\lambda(x) = e^{\lambda t} f_\lambda(x)$$

Remark : f_λ does not necessary belong to $L^2(\Gamma, d\mu)$ (in particular if $\lambda \in \mathrm{Sp}_{cont}(F^\mu \partial_\mu)$).

Interpretation of the Koopman modes

The Koopman modes represent the elementary coherent dynamical processes of the dynamical system on which we can decompose the evolution of all observable.



Two Koopman modes for aerodynamical turbulences exhibiting the shear-layer structures (left) and the wall structures (right) of the vortex.
C.W. Rowley *et al*, J. Fluid Mech. **641**, 115 (2009).

The Koopman modes are generalisations of the Fourier modes

Property

Let $(f_{\lambda_i})_{i\in\mathbb{N}}$ be a Koopman mode basis of $\mathcal{K}$ associated with simple Koopman values. We suppose that $\forall i \leq q$, $\Re\mathrm{e}\lambda_i = 0$; and that $\forall i > p$, $\Re\mathrm{e}\lambda_i < 0$. Then

$$\forall g \in \mathcal{K}, \forall i \leq p, \quad |f_{\lambda_i}(x)\rangle\langle f_{\lambda_i}|g\rangle = \lim_{T\to+\infty} \frac{1}{T}\int_0^T e^{-\lambda_i t} g(\varphi^t(x))dt$$

With $\lambda = 0$ and $f_0(x) = 1$ we refind the ergodic property (Corollary : a measure preserving system is ergodic iff all Koopman values are simple with unimodular Koopman modes).

With $\lambda_i \neq 0$, we find generalisations of the ergodic property.

The structure of the Koopman spectrum

Property

Let $\lambda_1, \lambda_2 \in \mathrm{Sp}(F^\mu \partial_\mu)$ *be two Koopman values associated with* f_{λ_1} *and* f_{λ_2}*, then* $\lambda_1 + \lambda_2 \in \mathrm{Sp}(F^\mu \partial_\mu)$ *associated with* $f_{\lambda_1+\lambda_2}(x) = f_{\lambda_1}(x) f_{\lambda_2}(x)$*. Moreover,* $\forall r \in \mathbb{R}$ *such that* $(f_{\lambda_1})^r \in \mathcal{C}^1(\Gamma)$*,* $r\lambda_1 \in \mathrm{Sp}(F^\mu \partial_\mu)$ *associated with* $f_{r\lambda_1}(x) = (f_{\lambda_1}(x))^r$*.*

Example : $\Gamma = \mathbb{S}^1$, $\varphi^t(\theta) = \theta + \omega t \mod 2\pi$ and $d\mu(\theta) = \frac{d\theta}{2\pi}$. The Koopman generator is $\omega \partial_\theta$ with spectrum $\mathrm{Sp}(\omega\partial_\theta) = \imath\omega\mathbb{Z}$ with $f_{\imath n \omega}(\theta) = e^{\imath n \theta}$. $f^r_{\imath n\omega} \in \mathcal{C}^1(\mathbb{S}^1)$ iff $r \in \mathbb{Z}$.

Composition of the Koopman spectrum

- $0 \in \mathrm{Sp}_d(F^\mu \partial_\mu)$, with degeneracy degree equal to the number of independent ergodic components of the flow in Γ, with $f_{0,i}$ constant on the i-th ergodic component, zero on the others.
- For all point x fixed, cyclic, on an invariant torus or on an attractor of the flow, the local Lyapunov exponents in $\mathbb{C}_{\Re\mathrm{e}<0}$ (and in $\imath\mathbb{R}$ for a fixed or cyclic point) are in $\mathrm{Sp}_d(F^\mu \partial_\mu)$, with $f_\lambda(x) = 0$ and $\vec{\nabla} f_\lambda \Big|_x$ the left Lyapunov eigenvector.
- For all cycle or invariant torus A of the flow, $\imath\omega\mathbb{Z} \subset \mathrm{Sp}_d(F^\mu \partial_\mu)$ for ω frequency of A, with $f_\lambda(x) \neq 0 \ \forall x \in A$.
- For all strange attractor or chaotic sea A of the flow, $\exists I \subset \imath\mathbb{R}^* \cap \mathrm{Sp}_{cont}(F^\mu \partial_\mu)$ with $f_\lambda(x) \neq 0 \ \forall x \in A$.
-
  - Some spectral values in $\mathbb{C}_{\Re\mathrm{e}\leq 0}$ such that $f_\lambda(x) = 0$ and $\vec{\nabla} f_\lambda \Big|_x = \vec{0}$ for all fixed or cyclic point x.
  - Some spectral values in $\mathbb{C}_{\Re\mathrm{e}>0}$ such that $f_\lambda(x) = +\infty$ for all point x fixed, cyclic, on an invariant torus or on an attractor.

3 Koopman approach of the networks
- Network dynamical systems
- Dynamical systems on physical networks

Dynamical systems on networks

X a network with a finite number of nodes numbered from 1 to N.
$\Gamma = \{1, ..., N\}$ discrete phase space.
$\varphi : \{1, ..., N\} \to \{1, ..., N\}$ node to node evolution map.
$\ell^\infty(\Gamma) \simeq \mathbb{C}^N$ with $\forall f \in \mathbb{C}^N$, $\|f\|_\infty = \sup_i |f_i|$.
$\ell^2(\Gamma) \simeq \mathbb{C}^N$ with $\forall f \in \mathbb{C}^N$, $\|f\|_2 = \sqrt{\sum_i |f_i|^2}$.
$\ell^1(\Gamma) \simeq \mathbb{C}^N$ with $\forall p \in \mathbb{R}^N_+$, $\|p\|_1 = \sum_i |p_i|$.
$\mathcal{T}, \mathcal{P} \in \mathfrak{M}_{N\times N}(\mathbb{C})$.

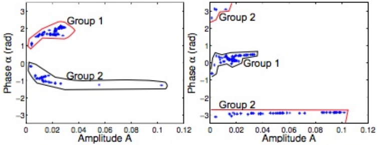
$$(\mathcal{T} f)_i = f_{\varphi(i)}$$

$$\mathcal{P} = \mathcal{T}^\dagger$$

Link with data networks

If φ describes a random surfer onto X (oriented) with $P(\varphi(i) = j)$ proportional to the number of links from i to j (and $P(\varphi(i) = j) = \frac{1}{N}$ $\forall j$, if i has no outgoing link), then $\mathcal{P} = \mathcal{T}^\dagger$ is the Google matrix of the network.

Partitioning power grids



(a) Coherency, Mode 2 (1.04 Hz) (b) Coherency, Mode 3 (1.72 Hz)

Fig. 3. Phase vs. amplitude plots for (a) Mode 2 (1.04 Hz) and (b) Mode 3 (1.72 Hz).

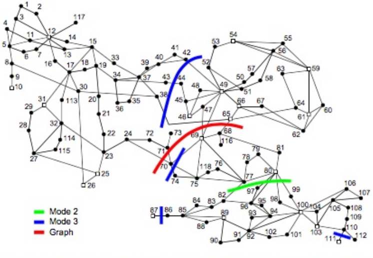


Fig. 4. Partitioning of the IEEE 118-bus test system according to Mode 2, Mode 3 and spectral graph theory. The cut-sets are indicated by colored lines.

Partitioning using coherent modes of the american electric power system in 1962 for a controlled islanding strategy in order to avoid cascading failures.

F. Raak *et al*, Innovative Smart Grid Technologies Conference, 2014 IEEE PES.

4 The Schrödinger-Koopman approach

- Classical-quantum mixing
- Quasienergies

Controlled quantum system

(Γ, φ^t, μ) is a classical dynamical system controlling a quantum system of Hilbert space $\mathcal{H}$:

$$\imath\hbar \frac{d\tilde{\psi}_{x_0}(t)}{dt} = H(\varphi^t(x_0))\tilde{\psi}_{x_0}(t), \quad \tilde{\psi}_{x_0} \in \mathcal{H}$$

$\Gamma \ni x \mapsto H(x) \in \mathcal{L}(\mathcal{H})$ is the controlled system Hamiltonian.

$$\rho(t) = \int_\Gamma |\tilde{\psi}_x(t)\rangle\langle\tilde{\psi}_x(t)|\rho_0(x)d\mu(x)$$

for $\rho_0 \in L^1_+(\Gamma, d\mu)$ the initial condition distribution.

Special case : $\rho_0(x) = \sum_{i=1}^N \delta(x - x_i)$ for a spin network :

$$\imath\hbar \frac{d\tilde{\psi}_{x_1,\dots,x_N}}{dt} = \left(\sum_{i=1}^N H_i(\varphi^t(x_i)) + H_{int}\right)\tilde{\psi}_{x_1,\dots,x_N}, \quad \tilde{\psi}_{x_1,\dots,x_N} \in (\mathbb{C}^2)^{\otimes N}$$

The Schrödinger-Koopman representation

$$\imath\hbar\frac{\partial\psi(x,t)}{\partial t} = H_K\psi(x,t), \quad \psi \in \mathcal{K}\otimes\mathcal{H}$$

$$H_K = -\imath\hbar F^\mu(x)\partial_\mu \otimes 1_{\mathcal{H}} + H(x)$$

Theorem

$$\tilde{\psi}_{x_0}(t) = \psi(\varphi^t(x_0),t)$$

$\tilde{\psi}_{x_0} \in \mathcal{H}$ is a solution of the Schrödinger equation.
$\psi \in \mathcal{K}\otimes\mathcal{H}$ is a solution of the Schrödinger-Koopman equation.

Classical-quantum entanglement

Theorem

Let $\psi \in \mathcal{K} \otimes \mathcal{H}$ *be the solution of the Schrödinger-Koopman equation with the initial condition* $\psi(x, t=0) = \sqrt{\rho_0(x)} \otimes \tilde{\psi}(t=0)$ *(*$\rho_0 \in L^1_+(\Gamma, d\mu)$*,* $\tilde{\psi}(t=0) \in \mathcal{H}$*), and* $\tilde{\psi}_x(t) \in \mathcal{H}$ *be the solution of Schrödinger equation with the initial condition* $\tilde{\psi}(t=0)$ *and* $x \in \Gamma$*.*

$$\begin{aligned} \rho(t) &= \int_\Gamma |\tilde{\psi}_x(t)\rangle\langle\tilde{\psi}_x(t)|\rho_0(x)d\mu(x) \\ &= \mathrm{tr}_{\mathcal{K}} |\psi(t)\rangle\!\rangle\langle\!\langle\psi(t)| \end{aligned}$$

Fake decoherence (averaging effect without entanglement) in the Schrödinger representation = True decoherence (non local entanglement between $\mathcal{H}$ and $\mathcal{K}$) in the Schrödinger-Koopman representation.

Quasienergies

$$(-\imath\hbar F^\mu(x)\partial_\mu + H(x))|a,x\rangle = \chi_a|a,x\rangle$$

$\{\chi_a\}_a = \mathrm{Sp}(H_K)$ is said to be the quasienergy spectrum (by generalisation of the Floquet theory corresponding to $\Gamma = \mathbb{S}^1$, $F(\theta) = \omega$, $d\mu(\theta) = \frac{d\theta}{2\pi}$).
NB : with an Hamiltonian system

$$H_K = \imath\hbar\{\mathscr{H}, \cdot\} + H(x)$$

$(\mathcal{K}, \imath\hbar\{\mathscr{H}, \cdot\})$ is a kind of (non-canonical) quantization of the classical dynamical system.

Quasienergy states

$$|a,x\rangle = f_{\lambda_a}(x)|Z\mu_{i_a},x\rangle$$

$\{f_\lambda\}_\lambda$ Koopman modes in $\mathcal{K}$, $\{|Z\mu_i,x\rangle\}_i$ basis of $\mathcal{H}$.

$\{|Z\mu_i,x\rangle\}_i$ are eigenvectors of $H(x)$ if x is a fixed point of the flow, or eigenvectors of the monodromy matrix of the Schrödinger equation if x is cyclic point.

Quasienergy and geometric phase

If the system is ergodic then

$$\begin{aligned} \chi_a &= \int_\Gamma \langle a, x | H(x) | a, x \rangle d\mu(x) - \imath\hbar \int_\Gamma F^\mu(x) \langle a, x | \partial_\mu | a, x \rangle d\mu(x) \\ &\sim \frac{1}{t} \int_0^t \langle a, \varphi^t(x_0) | H(\varphi^t(x_0)) | a, \varphi^t(x_0) \rangle dt - \frac{\imath\hbar}{t} \int_0^t i_{V(t)} A_a dt \end{aligned}$$

for t in the neighbourhood of infinity, $V(t) = F^\mu(x(t))\partial_\mu \in T\Gamma$ (tangent vector of the phase trajectory), and $A_a = \langle a, x | d | a, x \rangle \in \Omega^1\Gamma$ (geometric phase generator).

5 Bibliography

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