Adiabatic approximation for quantum dissipative systems: Formulation, topology, and superadiabatic tracking

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A generalized adiabatic approximation is formulated for a two-state dissipative Hamiltonian which is valid beyond weak dissipation regimes. The history of the adiabatic passage is described by superadiabatic bases as in the nondissipative regime. The topology of the eigenvalue surfaces shows that the population transfer requires, in general, a strong coupling with respect to the dissipation rate. We present, furthermore, an extension of the Davis-Dykhne-Pechukas formula to the dissipative regime using the formalism of Stokes lines. Processes of population transfer by an external frequency-chirped pulse-shaped field are given as examples.

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I. INTRODUCTION

Adiabatic processes in quantum mechanics, in which the wave function of a system follows instantaneous eigenstates, offer the great advantage of robustness of the corresponding process [1]. It is often termed as adiabatic passage when it concerns the coherent manipulation of atoms and molecules by an external pulse-shaped field [2]. Modern capabilities of laser systems have allowed this concept to be at the heart of many processes (see reviews [3,4] and more recent works such as, e.g., [5]).

Modern studies in quantum mechanics treat open systems which feature decoherence and more generally dissipative effects [6–8] with applications in quantum information science [9]. Extending adiabatic principles to such open quantum systems, treated using either a dissipative Schrödinger equation or a Lindblad equation, is a nontrivial issue. These systems are indeed described by non-Hermitian Hamiltonians (or Lindbladians) that are not necessarily diagonalizable and lead, in general, to complex eigenvalues corresponding to complex dynamical and geometrical phases [10–12].

Nondiagonalizable Hamiltonians lead to Jordan blocks in the most general situation which have to play the role of eigenspaces in the adiabatic approximation [13]. This has been rigorously proved in [14] through the use of superadiabatic renormalization techniques that allow suppression of the transitions between the Jordan blocks in the adiabatic limit.

However, the main obstacle to the existence of an adiabatic approximation is related to the appearance of complex eigenvalues: It is known that an adiabatic theorem in a standard form exists in general only when the dynamics follows the least dissipative state [11,12]. In the other case, when the dynamics follows the most dissipative state, only a weak dissipation allows preservation of its existence. The weak dissipation corresponds generally to the condition $\Gamma T \ll 1$, where Γ is the dissipation rate and T the characteristic time of adiabaticity, which limits exponentially growing terms. The nonexistence of an adiabatic theorem when this condition is not satisfied is critical since it prevents, in principle, its applicability to population transfer processes that are accompanied by crossing of the imaginary parts of the eigenvalues.

In this paper, we formulate the general solution of a dissipative two-state problem in the adiabatic limit, which is valid beyond weak dissipation regimes. We refer to this solution as a *generalized adiabatic approximation* for the dissipative two-state system. We denote the instantaneous eigenstates $|\varphi_j(\tau)\rangle$, $j = 1,2$, and the corresponding eigenvalues $\lambda_j(\tau)$, and one considers an initial connection with the state 1, $|\psi(-\infty)\rangle = |\varphi_1(-\infty)\rangle$. One key result is that the state solution after the interaction reads (with the phase condition for the eigenvectors satisfying the parallel transport)

$$|\psi(\tau)\rangle = a_1 e^{-iT\int_{-\infty}^{\tau}\lambda_1(\tau')d\tau'}|\varphi_1(\tau)\rangle + c_T(\tau)\, e^{-iT[\int_{-\infty}^{0}\lambda_1(\tau')d\tau'+\int_{0}^{\tau}\lambda_2(\tau')d\tau']}|\varphi_2(\tau)\rangle, \quad (1)$$

with the adiabatic and nonadiabatic coefficients, respectively, satisfying

$$a_1 = 1 + O(1/T), \quad (2a)$$

$$c_T(+\infty) = O(e^{-|\text{const}|\times T}). \quad (2b)$$

The eigenstate $|\varphi_1\rangle$ is referred to as the adiabatic state (which is expected to be "followed" during the dynamics), and $|\varphi_2\rangle$ as the nonadiabatic state. The use of the standard adiabatic bases in Eq. (1) allows the estimation of $c_T(\tau)$ only at large times $\tau \to +\infty$ as already stated. The superadiabatic formulation [15–19] allows one to derive an optimal superadiabatic basis in Eq. (1) (instead of the standard adiabatic basis) where the $c_T(\tau)$ coefficient can be well approximated at all times by a monotonic function given by the Gaussian error function from 0 to $c_T(+\infty)$. The error of the adiabatic coefficient can be improved using the optimal superadiabatic basis, in principle, as $a_1 = 1 + O(e^{-|\text{const}|\times T})$. Extending the derivation for the self-adjoint Hamiltonian [16,20–22] to the dissipative case, we can show the generic existence of this formulation using the formalism of Stokes lines and transition points [22]. This allows us to determine in principle the nonadiabatic coefficient $c_T(+\infty)$. This generalizes the Dykhne-Davis-Pechukas (DDP) formula [20,21] to the dissipative case (see also [23] for

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a determination of the nonadiabatic coefficients for specific models using perturbation theory).[1] It is shown here that, due to the complex phase terms in (1), even in the case of a well-satisfied adiabatic condition, the two components of the solution can be of the same order. It is even possible that the nonadiabatic component becomes larger, even much larger, than the adiabatic component. This is a consequence more precisely of the following observation: If the imaginary parts of the eigenvalues satisfy $\mathrm{Im}(\lambda_1) > \mathrm{Im}(\lambda_2)$ at all times, corresponding to a passage along the least dissipative adiabatic path, the second term in Eq. (1) is negligible at large times and one recovers a standard formulation of the adiabatic theorem, that is, along a single adiabatic state connected to the initial state. In the other case, $\mathrm{Im}(\lambda_2) > \mathrm{Im}(\lambda_1)$ (passage along the most dissipative adiabatic path), the second term can become larger than the first one from a certain *threshold time* [depending on the value of $c_T(+\infty)$]: *Adiabatic passage can allow continuous passage from one adiabatic state to the other*. This is, however, expected to occur at times when very little population is left in the system for the model we consider in this paper. One key message of this paper is that, in general, one has to consider the two terms in Eq. (1).

The paper is organized as follows. In the next section, we define the model and present the generalized mixing angle that allows the diagonalization of the system under the parallel transport conditions. Superadiabatic bases are defined. In Sec. III, we analyze the topology of the system: We classify all the possible paths in the parameter space that lead or do not lead to population transfer at the end of the interaction. We show that, in general, the population transfer can occur only in the situation of a coupling that is strong with respect to the dissipation rate. In Sec. IV, we sketch the construction of Eq. (1) in the adiabatic limit through examples of population transfer by an external frequency-chirped pulse-shaped field. The dynamics is also analyzed through a superadiabatic tracking. We conclude in Sec. V.

II. THE MODEL

A. Definition

Two-state models with a decaying state can be generically described by adding to the nonlossy Hamiltonian a loss rate $\Gamma > 0$ as a negative imaginary part to the corresponding diagonal term. We consider the upper excited state $|2\rangle$ as lossy. We consider, without loss of generality, that the population is initially in the ground state: $|\psi(-\infty)\rangle = |1\rangle$. The complete Hamiltonian, in the basis of the bare states $\{|1\rangle,|2\rangle\}$, reads (see, e.g., [24])

$$\hat{H} = \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega \\ \Omega & 2(\Delta - i\Gamma) \end{bmatrix}, \tag{3}$$

with the time-dependent real-valued parameters $\Delta \equiv \Delta(t/T)$ and $\Omega \equiv \Omega(t/T)$ referred to as the detuning (from the exact resonance) and the coupling (taken to be real for simplicity), respectively [2]. The loss can also be considered as time dependent in the general formulation: $\Gamma \equiv \Gamma(t/T)$. To extend the known formulations of adiabatic theorems [16,20–22] to non-Hermitian Hamiltonians, we first decompose the Hamiltonian into a part proportional to the identity and a traceless Hamiltonian: $\hat{H} = \frac{\hbar}{2}(\Delta - i\Gamma)\,\mathbb{1} + H$, with

$$H = \frac{\hbar}{2}\begin{bmatrix} -(\Delta - i\Gamma) & \Omega \\ \Omega & \Delta - i\Gamma \end{bmatrix}. \tag{4}$$

We analyze adiabatic passage for this traceless Hamiltonian H using the formalism of the Stokes lines and the transition points (see also [23] for a derivation using perturbation theory). The connection with the initial model is as follows:

$$\tilde{\psi}(t) = \exp\left\{-\frac{i}{2}\int_{-\infty}^{t} ds[\Delta(s/T) - i\Gamma(s/T)]\right\}\tilde{\phi}(t), \tag{5}$$

with $\tilde{\psi}(t)$ and $\tilde{\phi}(t)$ the solutions of the time-dependent Schrödinger equation with the Hamiltonian $\hat{H}$ and H, respectively.

It is convenient to introduce the characteristic duration of pulse T, which will be formally taken as $T \to \infty$ for mathematical considerations of the adiabatic theorem. We define the normalized time $\tau = t/T$ such that $\tilde{\psi}(t) = \psi(\tau)$ and $\tilde{\phi}(t) = \phi(\tau)$. We assume a pulsed coupling of peak Ω_0: $\Omega(\tau) = \Omega_0\Lambda(\tau)$ with $\Lambda(\pm\infty) = 0$.

B. Generalized mixing angle

We diagonalize the Hamiltonian (4) using the instantaneous transformation R associated with the complex mixing "angle" θ generalizing the real mixing angle of nonlossy systems:

$$R = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}, \quad \tan\theta = \frac{\Omega}{-\Delta + i\Gamma}, \tag{6}$$

with the branch chosen such that $0 \leqslant \mathrm{Re}(\theta) < \pi$ for positive Ω [$0 \leqslant \mathrm{Re}(\theta) < \pi/2$ for negative Δ, $\pi/2 \leqslant \mathrm{Re}(\theta) < \pi$ for positive Δ] and $\pi \leqslant \mathrm{Re}(\theta) < 2\pi$ for negative Ω [$\pi \leqslant \mathrm{Re}(\theta) < 3\pi/2$ for positive Δ, $3\pi/2 \leqslant \mathrm{Re}(\theta) < 2\pi$ for negative Δ]. It is associated with the right eigenvectors $|\varphi_+^{(R)}\rangle \equiv |\varphi_+\rangle = [\cos(\theta/2)\ \sin(\theta/2)]^t$, $|\varphi_-^{(R)}\rangle \equiv |\varphi_-\rangle = [-\sin(\theta/2)\ \cos(\theta/2)]^t$ (where the superscript t denotes the transpose), with the corresponding eigenvalues

$$\lambda_\pm = \pm\frac{1}{2}\lambda, \quad \lambda = \sqrt{(\Delta - i\Gamma)^2 + \Omega^2}, \tag{7}$$

and the left eigenvectors $|\varphi_\pm^{(L)}\rangle$ defined as $H^\dagger|\varphi_\pm^{(L)}\rangle = \lambda_\pm^*|\varphi_\pm^{(L)}\rangle$: $\langle\varphi_+^{(L)}| = [\cos(\theta/2)\quad \sin(\theta/2)]$ and $\langle\varphi_-^{(L)}| = [-\sin(\theta/2)\quad \cos(\theta/2)]$. The left and right eigenvectors allow one to define a biorthonormal basis with the scalar product: $\langle\varphi_i^{(L)}|\varphi_j^{(R)}\rangle = \delta_{ij}$. We can write the Schrödinger equation $i\partial_\tau|\phi\rangle = TH|\phi\rangle$ as $i\partial_\tau|\phi^{(1)}\rangle = TH^{(1)}|\phi^{(1)}\rangle$, with

$$TH^{(1)} = TR^{-1}HR - iR^{-1}\partial_\tau R = \begin{bmatrix} T\lambda_+ & i\gamma \\ -i\gamma & T\lambda_- \end{bmatrix}, \tag{8}$$

in terms of the adiabatic states $|\phi^{(1)}\rangle = R^{-1}|\phi\rangle$ and the nonadiabatic coupling $i\gamma \equiv i\gamma(\tau) \equiv i\partial_\tau\theta/2$. We remark that the use of this complex mixing angle is convenient since it allows very concise formulas for the normalization of the eigenvectors and for the nonadiabatic coupling, and more importantly it defines the phase of the eigenvectors under

[1] If one considers a loop in the parameter space, a Berry phase should be added (see, e.g., [23]).

parallel transport conditions: $\langle\varphi_\pm|\partial_\tau|\varphi_\pm\rangle = 0$; that is, no additional diagonal terms in the Hamiltonian $H^{(1)}$ (8) have to be considered.

C. Superadiabatic bases

One can construct superadiabatic bases, either as introduced by Berry [16], on which we expand the solution as a series, or by iterating the diagonalization procedure already presented [15,17,18]. The two procedures have been compared in [19] for nondissipative systems and show very similar results. Here we make the choice to describe and to use the iterative scheme for the example.

Denoting this first step with the index (0) (for the zeroth-order superadiabatic basis): $|\varphi_\pm^{(0)}\rangle \equiv |\varphi_\pm\rangle$, $\lambda_\pm^{(0)} \equiv \lambda_\pm$, $R_0 \equiv R$, $H^{(0)} \equiv H$, and after $n+1$ steps, one gets the Schrödinger equation $i\,\partial_\tau|\phi^{(n+1)}\rangle = T\,H^{(n+1)}|\phi^{(n+1)}\rangle$, with the state in the nth-order superadiabatic basis $|\phi^{(n+1)}\rangle = R_n^{-1}\cdots R_0^{-1}|\phi\rangle$ and $H^{(n+1)} = R_n^{-1}H^{(n)}R_n - i\,R_n^{-1}\partial_\tau R_n/T$.

With the use of superadiabatic bases instead of adiabatic bases, Eq. (1) has the same form but with the adiabatic eigenvectors and eigenvalues replaced with the optimal superadiabatic ones of order n:

$$|\psi(\tau)\rangle = a_1^{(n)} e^{-iT\int_{-\infty}^{\tau}\lambda_1^{(n)}(\tau')\,d\tau'}\left|\varphi_1^{(n)}(\tau)\right\rangle + c_T^{(n)}(\tau)\,e^{-iT[\int_{-\infty}^{0}\lambda_1^{(n)}(\tau')\,d\tau' + \int_0^{\tau}\lambda_2^{(n)}(\tau')\,d\tau']}\left|\varphi_2^{(n)}(\tau)\right\rangle, \tag{9}$$

where, if a single transition point z_0 is assumed for simplicity,

$$n = n_c \equiv \mathrm{Int}|T\delta_0(z_0)|, \tag{10a}$$

$$a_1^{(n)} = 1 + O(e^{-|\mathrm{const}|\times T}), \tag{10b}$$

$$c_T^{(n)}(\tau) \approx \frac{1}{2}\left\{1 + \mathrm{Erf}\left[\frac{\delta(\tau)}{\sqrt{2|\delta_0(z_0)|/T}}\right]\right\} c_T^{(n)}(+\infty), \tag{10c}$$

and

$$c_T^{(n)}(+\infty) = c_T(+\infty)e^{iT\int_{-\infty}^{0}[\lambda_1^{(n)}(\tau')-\lambda_1(\tau')]\,d\tau'} \times e^{iT\int_0^{+\infty}[\lambda_2^{(n)}(\tau')-\lambda_2(\tau')]\,d\tau'}, \tag{10d}$$

with

$$\delta_0(z) = \int_0^z \lambda(\tau')\,d\tau', \tag{11}$$

where Int is the integer part, and Erf is the Gaussian error function. The definition of the transition points for the dissipative system is given in Sec. IV.

III. TOPOLOGY OF ADIABATIC PASSAGE

We first characterize the topology of the system, generalizing the analysis of Ref. [25] to the dissipative case. We assume in this analysis an ideal and standard adiabatic evolution (i.e., neglecting the nonadiabatic transition, which corresponds to considering a time well before the threshold time). Figure 1 shows the eigenvalues of the Hamiltonian (3) as functions of the parameters $\Omega > 0$ and Δ (we would have symmetric identical surfaces for negative Ω). The Hamiltonian is not diagonalizable for $\Delta = 0$, $\Omega = \pm\Gamma$, which corresponds to the two singularities of the square root branch of the eigenvalues. This is shown in Fig. 2, where the absolute value of the

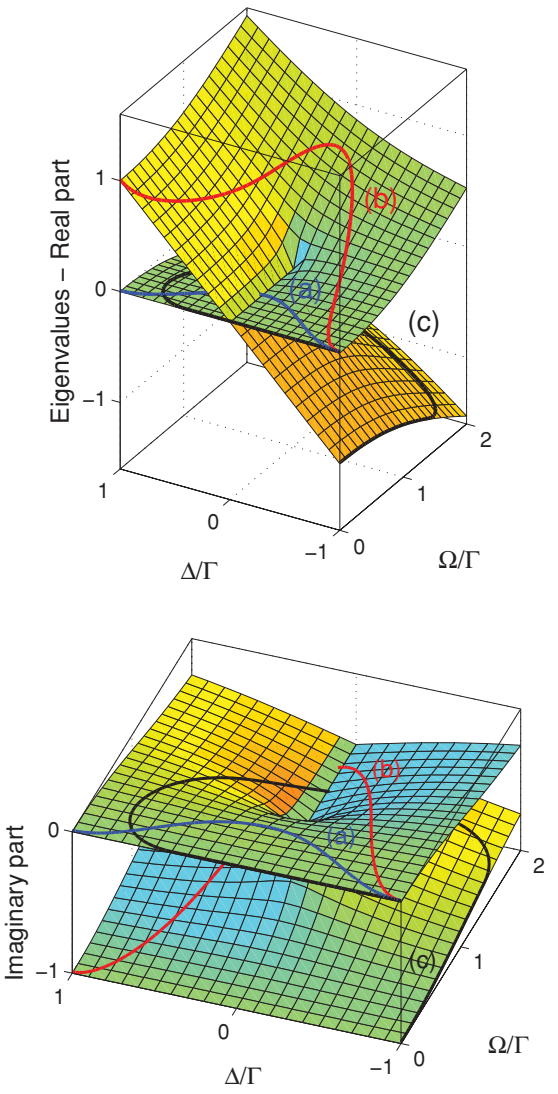

FIG. 1. (Color online) Real and imaginary parts of the eigenvalue surfaces (in units of Γ) of Hamiltonian (3) as functions of Ω/Γ and Δ/Γ. Two paths, (b) and (c), which surround the singularity (located at $\Omega = \Gamma$, $\Delta = 0$), and one path, (a), which does not, are shown. Note that the nonphysical discontinuity of the imaginary surfaces that takes place for $\Delta = 0$, $\Omega < \Gamma$, has been removed so that path (a), following the upper surface continuously, features the actual adiabatic dynamics.

difference between the two eigenvalues of (4), $\pm|\lambda|/2$, are displayed. The singularity for $\Delta = 0$, $\Omega = +\Gamma$, is exhibited as a hole in the surfaces. One can define a branch cut between these two singularities, that is, for $-\Gamma \leqslant \Omega \leqslant \Gamma$ at $\Delta = 0$: It corresponds here to equal real parts of the eigenvalues. For $\Omega > \Gamma$ at $\Delta = 0$, the imaginary parts of the eigenvalues are equal. The chosen branch that defines the eigenstates corresponds to the Riemann sheet defined as $\sqrt{z^2} = z$ (with $z = \Omega$). This leads to the labeling of the eigenvalues such that the real part of the upper (lower) surface corresponds to λ_+ (λ_-) (see a discussion of the labeling of such surfaces in [26]). We remark that such a labeling implies a discontinuity of the imaginary part of the eigenvalues when one crosses the branch cut. We do not show this discontinuity in Fig. 1, since it does

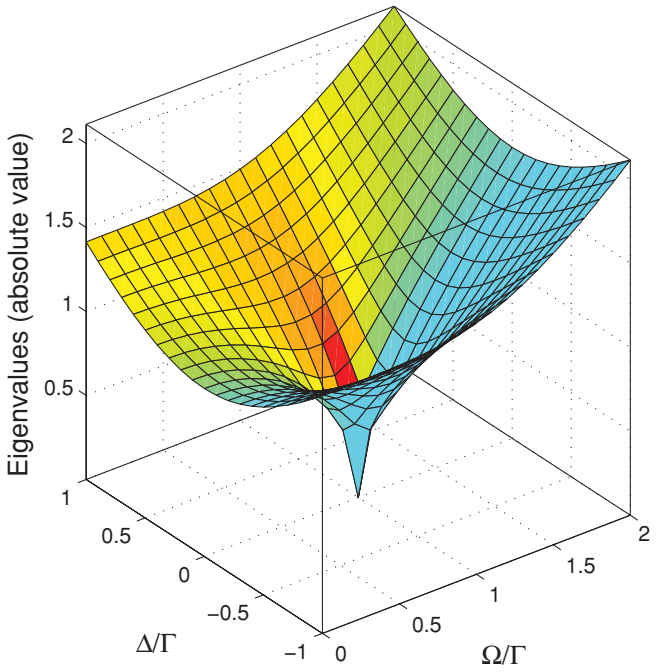

FIG. 2. (Color online) Absolute value of the difference λ of the eigenvalue surfaces (in units of Γ) of Hamiltonian (4) as functions of Ω/Γ and Δ/Γ.

not correspond to the actual adiabatic dynamics, as shown here. In Fig. 1, state $|1\rangle$ corresponds to the horizontal line of energy 0 in the plane $\Omega = 0$, and state 2 to the line Δ for the real part and the horizontal line $-i\Gamma$ for the imaginary part in the plane $\Omega = 0$.

Figure 1 allows one to determine *topologically different adiabatic dynamics* depending on whether the path surrounds one, two, or no singularities. When the path is such that $\Omega_0 < \Gamma$ [i.e., not surrounding the singularity; see path (a) as an example], the adiabatic dynamics crosses the branch cut (at $\Delta = 0$) and thus "jumps" from the surface λ_+ (if it is assumed to be the initial one) to λ_-. This jump appears for the real part as a path going through the crossing. For the imaginary part this jump is not apparent, since we have removed the discontinuity between the two surfaces. Mathematically this jump corresponds to a passage to the lower Riemann sheet for which $\sqrt{z^2} = -z$. This physically signifies an absence of transition as shown by the corresponding path (a) in Fig. 1. When $\Omega_0 > \Gamma$, such that the path surrounds the singularity $\Omega = \Gamma$, $\Delta = 0$ [see path (b)], the dynamics corresponds to a transition from state $|1\rangle$ to state $|2\rangle$, accompanied by *a crossing of the imaginary part of the eigenvalues when* $\Delta = 0$. Figure 1 also shows a path labeled (c) as a loop in the parameter space surrounding the singularity. Considering also negative Ω, one can characterize a path surrounding the two trajectories, leading to an absence of transition (not shown).

IV. ADIABATIC APPROXIMATION AND SUPERADIABATIC TRACKING

In the adiabatic limit $T \to \infty$, one could expect the existence of a standard adiabatic theorem since in that case the nonadiabatic coupling with respect to the gap between the eigenvalues is small: $\epsilon := |\gamma|/T|\lambda_+ - \lambda_-| \to 0$. This is indeed true in the nonlossy Hermitian case but *not*, in general, for a lossy system [11,12]. To show it, we expand the state solution as

$$|\phi(\tau)\rangle = \sum_{j=+,-} b_j(\tau)|\varphi_j(\tau)\rangle, \tag{12}$$

with

$$b_j(\tau) = a_j(\tau)\, e^{-iT\int_{-\infty}^{\tau}\lambda_j(s)\,ds}. \tag{13}$$

The equations read

$$\partial_\tau a_+(\tau) = \gamma(\tau)\, e^{iT\delta(\tau)} a_-(\tau), \tag{14a}$$

$$\partial_\tau a_-(\tau) = -\gamma(\tau)\, e^{-iT\delta(\tau)} a_+(\tau). \tag{14b}$$

with

$$\delta(\tau) = \int_{-\infty}^{\tau} [\lambda_+(\tau') - \lambda_-(\tau')]\, d\tau' = \int_{-\infty}^{\tau} \lambda(\tau')\, d\tau', \tag{15}$$

and can be rewritten as the equivalent Volterra equations:

$$a_+(\tau) = a_+(-\infty) + \int_{-\infty}^{\tau} \gamma(\tau')\, e^{iT\delta(\tau')} a_-(\tau')\, d\tau', \tag{16a}$$

$$a_-(\tau) = a_-(-\infty) - \int_{-\infty}^{\tau} \gamma(\tau')\, e^{-iT\delta(\tau')} a_+(\tau')\, d\tau'. \tag{16b}$$

In practice, using, for simplicity, a parametrization such that $\Omega^2(\tau) + \Delta^2(\tau) = \Omega_0^2$ (which corresponds to a circular path in the parameter space, leading to the initial and final detuning in absolute value $|\Delta_0| = \Omega_0$; see [27]), one gets a small ϵ for

$$T\left(\Omega_0^2 - \Gamma^2\right)^{3/2} \gg \Omega_0^2. \tag{17}$$

Superadiabatic bases are more adapted to the tracking of the solution, since they induce similar equations to (14) but with a reduced nonadiabatic coupling [in principle, exponentially small as $O(e^{-|\text{const}|\times T})$ for high-order bases].

A. Noncrossing models

We first analyze *noncrossing models* which correspond to the situation when the imaginary parts of the eigenvalues do not cross. In that case, we can consider, without loss of generality, that

$$\text{Im}(\lambda_-) < \text{Im}(\lambda_+) \leqslant 0 \tag{18}$$

for all times τ; that is, $\text{Im}[\delta(\tau)] > 0$. This means that the state $|\varphi_+\rangle$ is less dissipative than the state $|\varphi_-\rangle$.

1. Connection to the least dissipative eigenvalue

We first consider an initial condition connected to the state $+$, $|\phi(-\infty)\rangle = |1\rangle = |\varphi_+(-\infty)\rangle$, which is achieved when $\Delta(-\infty) = -\Delta_0 < 0$ (i.e., $\Delta_0 > 0$), corresponding to $a_+(-\infty) = b_+(-\infty) = 1, a_-(-\infty) = b_-(-\infty) = 0$. It thus leads to an adiabatic following along the least dissipative eigenstate $|\varphi_+(\tau)\rangle$.

Integrating Eqs. (16) once by parts, and evaluating an upper bound for the norms $|a_+(\tau)|$ and $|a_-(\tau)|$, we get

$$a_+(\tau) = 1 + O(1/T), \tag{19a}$$

$$a_-(\tau) = O(e^{T\text{Im}[\delta(\tau)]}/T), \tag{19b}$$

while, in the adiabatic limit $T \to \infty$, $a_+(\tau) \to 1$, $|a_-(\tau)|$ grows exponentially. However, concerning the coefficients $b_\pm$, one can calculate at late times that

$$b_-(\tau)/b_+(\tau) = O(e^{-T\mathrm{Im}[\delta(\tau)]}) \tag{20}$$

due to the stronger dissipative exponent in front of the coefficient a_-. A standard adiabatic theorem can thus be stated along the least dissipative eigenvalue [11,12]: $|\phi(\tau)\rangle \simeq e^{-iT\int_{-\infty}^{\tau}\lambda_+(\tau')\,d\tau'}|\varphi_+(\tau)\rangle$.

More precisely, extending to the dissipative case the technique of Stokes lines and transition points as rigorously established for the self-adjoint case in Ref. [22], one obtains, for the coefficient $c_T(+\infty)$ defined in Eq. (1),

$$c_T(+\infty) = \sum_{j=0}^{N-1} e^{-i\theta_j} e^{-iT\delta_0(z_j)} + O(e^{T\mathrm{Im}[\delta_0(z_0)]}/T), \tag{21}$$

with

$$\delta_0(z) = \int_0^z \lambda(\tau')\,d\tau', \tag{22}$$

where the N transition point(s) z_j lies on the first Stokes line z in the *lower* complex plane, defined as

$$\mathrm{Im}[\delta_0(z)] = \mathrm{Im}[\delta_0(z_0)] = \mathrm{Im}[\delta_0(z_{j=1,N-1})] < 0, \tag{23}$$

and θ_j is a geometrical phase [22,23,28]. In the case considered here, for which Ω is real, θ_j is 0 or π. The first Stokes line corresponds to the one giving the smallest $\mathrm{Im}[\delta_0(z_0)]$ in absolute value (which gives the largest contribution). This result corresponds to the approximation

$$|\phi(\tau)\rangle \approx e^{-iT\int_{-\infty}^{\tau}\lambda_+(\tau')\,d\tau'}|\varphi_+(\tau)\rangle + c_T(\tau)\,e^{-iT[\int_{-\infty}^{0}\lambda_+(\tau')\,d\tau'+\int_0^{\tau}\lambda_-(\tau')\,d\tau']}|\varphi_-(\tau)\rangle, \tag{24}$$

which is (1) in the original model (3) with the labeling $1 \leftrightarrow +$ for the eigenvectors and $\lambda_1 = \frac{1}{2}(\Delta - i\Gamma) + \lambda_+$, $\lambda_2 = \frac{1}{2}(\Delta - i\Gamma) + \lambda_-$.

2. Connection to the most dissipative eigenvalue

In the opposite situation, when $|\phi(-\infty)\rangle = |1\rangle = |\varphi_-(-\infty)\rangle$, which is achieved when $\Delta(-\infty) = \Delta_0 > 0$, corresponding to $a_-(-\infty) = b_-(-\infty) = 1, a_+(-\infty) = b_+(-\infty) = 0$, we cannot state a standard adiabatic theorem along the most dissipative eigenstate in general, in the sense that the single most dissipative eigenstate is not "followed" during the complete dynamics. Following the same procedure as before, we indeed get

$$a_-(\tau) = 1 + O(1/T), \tag{25a}$$

$$a_+(\tau) = O(e^{-T\mathrm{Im}[\delta(\tau)]}/T), \tag{25b}$$

and

$$b_+(\tau)/b_-(\tau) = O(e^{T\mathrm{Im}[\delta(\tau)]}). \tag{26}$$

The latter equation signifies that $|b_-(\tau)|$, much larger than $|b_+(\tau)|$ at early times, crosses it at a threshold (late) time, before becoming exponentially smaller. These arguments prevent the construction of a standard adiabatic theorem in general along the most dissipative eigenstate. It can be established only in the limit of a weak dissipation $T\mathrm{Im}[\delta(\tau)] \ll 1$. We remark that the coefficient corresponding to the nonadiabatic term $a_+(\tau)$ decreases exponentially. The absence of the general adiabatic theorem is thus due only to the competition between the dissipative phases associated with the two eigenstates.

Using the Stokes lines technique, one derives, similarly to Eq. (21),

$$c_T(+\infty) = \sum_{j=0}^{N-1} e^{-i\theta_j} e^{iT\delta_0(z_j)} + O(e^{-T\mathrm{Im}[\delta_0(z_0)]}/T), \tag{27}$$

where the N transition point(s) z_j lie on the first Stokes line z in the *upper* complex plane defined as

$$\mathrm{Im}[\delta_0(z)] = \mathrm{Im}[\delta_0(z_0)] = \mathrm{Im}[\delta_0(z_{j=1,N-1})] > 0. \tag{28}$$

This leads to the approximation

$$|\phi(\tau)\rangle \approx e^{-iT\int_{-\infty}^{\tau}\lambda_-(\tau')\,d\tau'}|\varphi_-(\tau)\rangle + c_T(\tau)\,e^{-iT[\int_{-\infty}^{0}\lambda_-(\tau')\,d\tau'+\int_0^{\tau}\lambda_+(\tau')\,d\tau']}|\varphi_+(\tau)\rangle, \tag{29}$$

which is (1) in the original model (3), with the labeling $1 \leftrightarrow -$ for the eigenvectors and $\lambda_1 = \frac{1}{2}(\Delta - i\Gamma) + \lambda_-$, $\lambda_2 = \frac{1}{2}(\Delta - i\Gamma) + \lambda_+$.

3. Particular case: Even coupling and constant detuning

In the case of a coupling of even symmetry, $\Omega(-\tau) = \Omega(\tau)$, and of a constant detuning, we can simplify the phase in Eq. (1) for large final times $\tau_f = -\tau_i \to +\infty$ opposite to the initial time τ_i: $\int_{\tau_i}^{0}\lambda_1(\tau')\,d\tau' + \int_0^{\tau_f}\lambda_2(\tau')\,d\tau' = \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau'[\Delta - i\Gamma(\tau')]$. The asymptotic solution reads, in that case,

$$|\psi(\tau_f)\rangle = [1 + O(1/T)]e^{-iT\int_{\tau_i}^{\tau_f}\lambda_1(\tau')\,d\tau'}|\varphi_1(\tau_f)\rangle + c_T(\tau_f)\,e^{-T\frac{i}{2}\int_{\tau_i}^{\tau_f} d\tau'[\Delta - i\Gamma(\tau')]}|\varphi_2(\tau_f)\rangle. \tag{30}$$

B. Crossing models

If one considers an adiabatic passage in a *crossing model* leading to a transition from the ground state to the lossy excited state, and thus corresponding to a crossing of the imaginary parts of the eigenvalues, we have to extend the two preceding results.

1. Initial connection to the least dissipative eigenvalue

If the initial condition is connected to the state $+$, $|\phi(-\infty)\rangle = |1\rangle = |\varphi_+(-\infty)\rangle$, that corresponds to the initial following along the least dissipative eigenstate (before the eigenvalues cross), we obtain [with $1 \leftrightarrow +$ in Eq. (1)]

$$a_+ = 1 + O(1/T), \tag{31a}$$

$$c_T(+\infty) = \sum_{j=0}^{N-1} e^{-i\theta_j} e^{-iT\delta_0(z_j)} + O(e^{T\mathrm{Im}[\delta_0(z_0)]}/T), \tag{31b}$$

where the N transition point(s) z_j lies on the first Stokes line z in the *lower* complex plane. This is similar to the situation

of the following along the least dissipative eigenstate for the noncrossing model.

2. *Initial connection to the most dissipative eigenvalue*

In the opposite situation, when the initial condition is connected to the state $-$, $|\phi(-\infty)\rangle = |1\rangle = |\varphi_-(-\infty)\rangle$, that corresponds to the initial following along the most dissipative eigenstate (before the eigenvalues cross), we obtain [with $1 \leftrightarrow -$ in Eq. (1)]

$$a_- = 1 + O(1/T), \tag{32a}$$

$$c_T(+\infty) = \sum_{j=0}^{N-1} e^{-i\theta_j} e^{iT\delta_0(z_j)} + O(e^{-T\mathrm{Im}[\delta_0(z_0)]}/T), \tag{32b}$$

where the N transition point(s) z_j lies on the first Stokes line z in the *upper* complex plane (similarly to the situation of the following along the most dissipative eigenstate for the noncrossing model).

3. *Numerical illustration: The dissipative Allen-Eberly model*

A typical dynamics of a crossing model is shown in Fig. 3, here for the Allen-Eberly model [29] to which a constant dissipation is added. We consider the case where the dynamics first follows the least dissipative eigenstate $|\varphi_+(-\infty)\rangle$: The standard adiabatic theorem applies as long as $\epsilon \ll 1$ and until the imaginary parts of the eigenvalues cross. As already discussed and as shown in Fig. 3, this regime corresponds to $a_+(\tau) \simeq 1$ and $|b_-(\tau)| \ll |b_+(\tau)|$. During this part of the dynamics, the other coefficient $|c_T(\tau)|$ starts growing exponentially. As expected, the use of superadiabatic bases of higher orders (here shown up to $n = 3$) allows the removal of this growth.

After the crossing of the imaginary part of the eigenvalues, the dynamics follows the most dissipative eigenstate. The main point here is that, as expected from the previous analysis, the coefficient $|c_T(\tau)|$ goes to a small quantity for $\tau \to \infty$, corresponding to the nonadiabatic correction, and that $|a_+(\tau)|$ stays localized around the value 1 oscillating. The use of superadiabatic bases allows a dramatic reduction of the oscillations, which become unnoticeable.

The standard adiabatic theorem does not apply near and beyond the crossing of the coefficients $|b_+(\tau)|$ and $|b_-(\tau)|$. We remark that the superadiabatic bases of higher orders allow a larger separation between the coefficients $|b_+(\tau)|$ and $|b_-(\tau)|$ before their crossing (whose position is not shifted), which permits a better standard adiabatic approximation than the one expected from the zeroth order.

We can determine for this example the asymptotic value of $|c_T|^2$ using formula (31b). We have one transition point on the first Stokes line in this case for $\Delta_0 > \Omega_0$. We obtain $|c_T(+\infty)|^2 \approx 0.016$, which corresponds to the value obtained numerically (expressed in the development with respect to the zeroth superadiabatic basis). We have also plotted in Fig. 3 the coefficient $c_T^{(n)}(\tau)$ from Eqs. (10c) and (10d): It is almost undistinguishable from the numerical values.

One notes in Fig. 3 that, for the subsequent superadiabatic orders, the coefficient $|c_T^{(n)}(+\infty)|^2$ varies slightly, following Eq. (10d). This behavior, which does not occur in the nondissipative problem, is generic in the dissipative case when one uses the superadiabatic iterative scheme. The superadiabatic bases induce indeed a (small) correction of the eigenvalues at each order (only when the interaction is on). Equation (1), rewritten with the superadiabatic bases instead of the adiabatic bases, that is, as Eq. (9), thus shows that, at late times, when the interaction is off, that is, when the superadiabatic eigenvectors are identical to the canonical bases (for any order), the integration of the eigenvalues necessarily implies a modification of the coefficient $c_T^{(n)}(+\infty)$ with respect to $c_T(+\infty)$ following (10d) to recover the same coefficient in front of the eigenvector.

This example confirms the general approximation (1) we have stated.

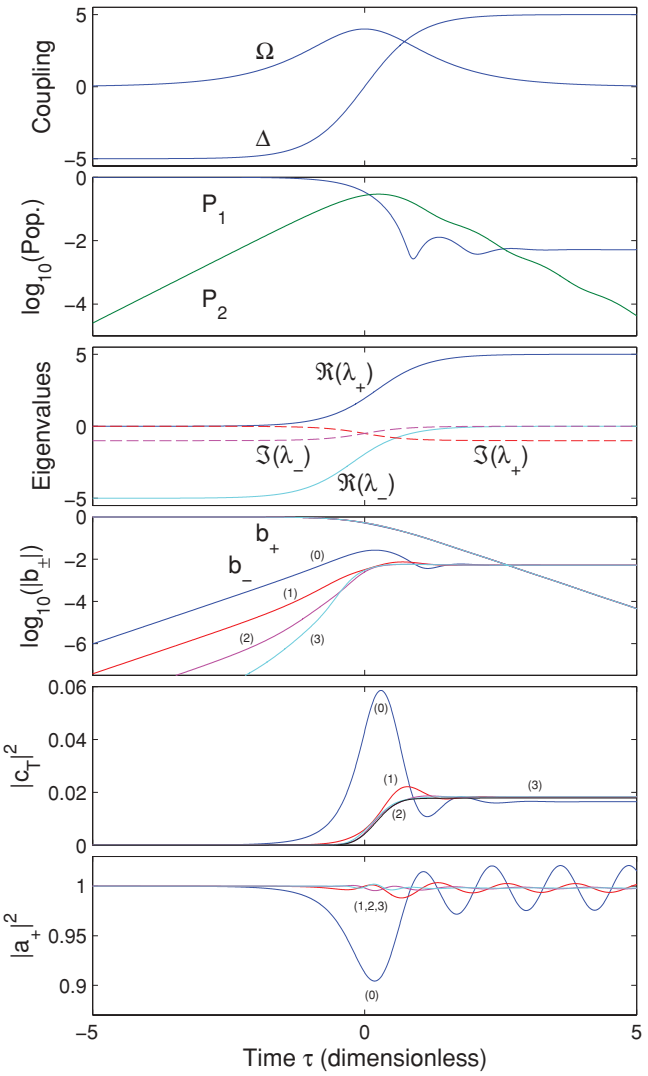

FIG. 3. (Color online) Dynamics for the dissipative Allen-Eberly model $\Lambda(\tau) = \mathrm{sech}(\tau)$, $\Delta(\tau) = \Delta_0 \tanh(\tau)$, $\Omega_0 T = 4$, $\Delta_0 T = 5$, and $\Gamma T = 1$, corresponding to the type of path (a) surrounding the singularity in Fig. 1. From top to bottom: Time-dependent coupling and detuning, populations in the bare states $P_j = |\langle j|\psi\rangle|^2$, instantaneous eigenvalues (real parts, solid lines; imaginary parts, dashed lines), and coefficients $b_\pm$, c_T, and a_+. Numbers in parentheses, from (0) to (3), indicate the order of the superadiabatic iteration; see text.

V. DISCUSSION AND CONCLUSION

Our analysis has allowed the formulation of a general adiabatic approximation in a dissipative crossing or noncrossing model. The occurrence of a transition requires a strong coupling in the sense that $\Omega_0 > \Gamma$, as shown by the topology of the eigenvalue surfaces. Determination of the precise conditions that allow approximation (1) is in progress, and this will be the subject of a forthcoming paper.

Our analysis supports the generalized Berry phases in dissipative models (see, e.g., [30]). Our result is also applicable in the case of time-dependent dissipation rates. They can be encountered, for instance, in a process of population transfer from the ground state of an atom to an excited state that is coupled to a continuum (see, e.g., [31]).

A further important issue is whether the techniques presented in this paper, namely, the topology of the eigensurfaces, superadiabatic bases, and generalized DDP formula, are applicable to a dissipative system with more than two states. This question arises in the frame of both the Schrödinger equation and the Lindblad equation. The latter corresponds to an effective system of three components on the Bloch sphere for a two-state system that leads, in general, to complex adiabatic eigenvalues. The simplest extension concerns the analysis of the passage using eigensurfaces (as functions of the field parameters) decomposed as a real part and an imaginary part. This analysis has been performed in multistate systems without dissipation (see, e.g., [32]). The main different feature for such multistate systems with complex eigenvalues is the presence of zones in the parameter space where the Hamiltonian is not diagonalizable [13]. The use of superadiabatic techniques has been applied in that case to suppress the transitions between the Jordan blocks in the adiabatic limit [14]. We remark that it would be of interest to find a physical system that features such a particular property and to describe its influence on the dynamics of population transfer.

More generally, using superadiabatic techniques in an N-level system ($N > 2$) to determine a common superadiabatic basis that would be optimal for the transitions, which are multiple in the generic case, is questionable. This has, however, been successfully applied for the example of the nonlossy stimulated Raman adiabatic process in a Λ system [19]. The presence of additional loss, as, for instance, from the upper state as spontaneous emission outside the considered three-state system, has not been considered.

The extension of the DDP formula to nonlossy N-level systems was described in the original paper by Hwang and Pechukas [21]. It has been shown to be complicated and not generically solvable due to numerous crossings in the complex plane. Only specific symmetries in the Hamiltonian allow this extension [33–37]. In the simplest case, the result can be interpreted as local Landau-Zener analysis of the consecutive avoided crossings between pairs of levels assumed to be separated, which do not involve interfering paths, such that the final probability is the product of the probabilities corresponding to the consecutive avoided crossings. The presence of an additional dissipation is expected to lead to a similar analysis and to similar results. Calculations in a two-state system submitted to a dephasing loss through the Lindblad equation, and leading to an effective three-component system with specific symmetries, have given results compatible with such an extension of the DDP formula [38]. Such a system could be a physical candidate for application of the technique of superadiabatic bases and a DDP formula to lossy N-level systems.

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[1] A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1962).
[2] B. W. Shore, *The Theory of Coherent Atomic Excitation* (Wiley, New York, 1990); Acta Phys. Slovaca **58**, 243 (2008).
[3] N. V. Vitanov, T. Halfmann, B. W. Shore, and K. Bergmann, Annu. Rev. Phys. Chem. **52**, 763 (2001).
[4] S. Guérin and H. R. Jauslin, Adv. Chem. Phys. **125**, 147 (2003).
[5] E. A. Shapiro, A. Pe'er, J. Ye, and M. Shapiro, Phys. Rev. Lett. **101**, 023601 (2008).
[6] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, UK, 2002).
[7] C. W. Gardiner and P. Zoller, *Quantum Noise* (Springer, Berlin, 2004).
[8] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 2008).
[9] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[10] J. C. Garrison and E. M. Wright, Phys. Lett. A **128**, 177 (1988).
[11] A. Kvitsinsky and S. Puttermann, J. Math. Phys. **32**, 1403 (1991).
[12] G. Nenciu and G. Rasche, J. Phys. A **25**, 5741 (1992).
[13] M. S. Sarandy and D. A. Lidar, Phys. Rev. Lett. **95**, 250503 (2005); Phys. Rev. A **71**, 012331 (2005).
[14] A. Joye, Commun. Math. Phys. **275**, 139 (2007).
[15] M. V. Berry, Proc. R. Soc. London A **414**, 31 (1987).
[16] M. V. Berry, Proc. R. Soc. London A **429**, 61 (1990).
[17] G. Nenciu, Commun. Math. Phys. **152**, 479 (1993).
[18] A. Joye and C.-E. Pfister, J. Math. Phys. **34**, 454 (1993).
[19] K. Drese and M. Holthaus, Eur. Phys. J. D **3**, 73 (1998).
[20] A. M. Dykhne, Zh. Eksp. Teor. Fiz. **41**, 1324 (1962) [Sov. Phys. JETP **14**, 941 (1962)]; J. P. Davis and P. Pechukas, J. Chem. Phys. **64**, 3129 (1976).
[21] J.-T. Hwang and P. Pechukas, J. Chem. Phys. **67**, 4640 (1977).
[22] A. Joye, H. Kuntz, and Ch.-Ed. Pfister, Ann. Phys. (NY) **208**, 299 (1991); A. Joye, G. Mileti, and Ch.-Ed. Pfister, Phys. Rev. A **44**, 4280 (1991).
[23] R. Schilling, M. Vogelsberger, and D. A. Garanin, J. Phys. A **39**, 13727 (2006).
[24] V. M. Akulin and W. P. Schleich, Phys. Rev. A **46**, 4110 (1992); N. V. Vitanov and S. Stenholm, *ibid.* **55**, 2982 (1997).

[25] S. Guérin, L. P. Yatsenko, and H. R. Jauslin, Phys. Rev. A **63**, 031403(R) (2001); L. P. Yatsenko, S. Guérin, and H. R. Jauslin, *ibid.* **65**, 043407 (2002).
[26] D. Viennot, G. Jolicard, and J. P. Killingbeck, J. Phys. A **41**, 145303 (2008).
[27] S. Guérin, S. Thomas, and H. R. Jauslin, Phys. Rev. A **65**, 023409 (2002).
[28] M. V. Berry, Proc. R. Soc. London A **430**, 405 (1990).
[29] L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Dover, New York, 1987).
[30] A. I. Nesterov and S. G. Ovchinnikov, Phys. Rev. E **78**, 015202(R) (2008).
[31] L. P. Yatsenko, V. R. Romanenko, B. W. Shore, T. Halfmann, and K. Bergmann, Phys. Rev. A **71**, 033418 (2005).
[32] S. Thomas, S. Guérin, and H. R. Jauslin, Phys. Rev. A **71**, 013402 (2005).
[33] C. E. Caroll and F. T. Hioe, J. Phys. A **19**, 1151 (1986).
[34] S. Brundobler and V. Elser, J. Phys. A **26**, 1211 (1993).
[35] A. Joye and C.-E. Pfister, Asymptotic Anal. **23**, 91 (2000).
[36] C. Fermanian-Kammerer and V. Rousse, e-print arXiv: 0904.0211v2 [math.AP].
[37] S. S. Ivanov and N. V. Vitanov, Phys. Rev. A **77**, 023406 (2008).
[38] X. Lacour, S. Guérin, and H. R. Jauslin, Phys. Rev. A **78**, 033417 (2008).