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Principal bundle structure of quantum adiabatic dynamics with a Berry phase which does not commute with the dynamical phase

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A geometric model is proposed to describe the Berry phase phenomenon when the geometric phase does not commute with the dynamical phase. The structure used is a principal composite bundle in which the adiabatic transport appears as a horizontal lift. The formulation is applied to a simple quantum dynamical system controlled by two lasers. © *2005 American Institute of Physics.*
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I. INTRODUCTION

Principal bundle theory is a classic tool of modern theoretical physics. The notation (P,M,G,π) will be used throughout this paper to designate a principal bundle on the right-hand side with base space M, total space P, structure group G, and projection π. First we quote an important result of principal bundle theory. Suppose that P is endowed with a connection described by the gauge potential A^σ associated to the local section $\sigma \in \Gamma(M,P)$. Let $\mathcal{C}$ be a curve in M, parametrized by the function $[0,1] \ni t \mapsto \gamma(t) \in M$. Then the horizontal lift of $\mathcal{C}$ at the point $\sigma(\gamma(0))$ is given by

$$P \ni p(t) = \sigma(\gamma(t))\mathbb{P}e^{-\int_0^t A^\sigma(\gamma(t))}, \tag{1}$$

where $\mathbb{P}$ is the path-ordering operator and $\mathbb{P}e^{-\int_0^t A^\sigma(\gamma(t))} \in G$ acts on P by the group canonical right action.

In 1984, Berry[1] proved, in the context of the standard adiabatic approximation, that the wave function of a quantum dynamical system takes the form

$$\psi(t) = e^{-\imath\hbar^{-1}\int_0^t E_a(\vec{R}(t'))\mathrm{d}t' - \int_0^t \langle a,\vec{R}(t')|\partial_{t'}|a,\vec{R}(t')\rangle \mathrm{d}t'}|a,\vec{R}(t)\rangle, \tag{2}$$

where E_a is a nondegenerate instantaneous eigenvalue isolated from the rest of the Hamiltonian spectrum with instantaneous eigenvector $|a,\vec{R}(t)\rangle$ and $\vec{R}$ is a set of classical control parameters used to model the time-dependent environment of the system. The set of all configurations of $\vec{R}$ is supposed to form a $\mathcal{C}^\infty$-manifold $\mathcal{M}$. The important result is the presence of the extra phase term, called the Berry phase $e^{-\int_0^t \langle a,\vec{R}(t')|\partial_{t'}|a,\vec{R}(t')\rangle \mathrm{d}t'}$. Simon[2] later found the mathematical structure which models the Berry phase phenomenon, namely a principal bundle with base space $\mathcal{M}$ and with structure group U(1). If we eliminate the dynamical phase by a gauge transformation which involves redefining the eigenvector at each time, then the expression (2) is the horizontal lift of the curve $\mathcal{C}$ described by $t \mapsto \vec{R}(t)$ with the gauge potential $A = \langle a,\vec{R}|\mathrm{d}_\mathcal{M}|a,\vec{R}\rangle$. If $\mathcal{C}$ is closed then the Berry phase $e^{-\oint_\mathcal{C} A} \in \mathrm{U}(1)$ is the holonomy of the horizontal lift.

In 1987, Aharonov and Anandan[3] proved that geometric phases such as the Berry phase are not solely attached to the adiabatic approximation but appear in a more general context. Let

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$t \mapsto \psi(t)$ be a wave function such that $\psi(T)=e^{\imath\phi}\psi(0)$ and $H(t)$ be the Hamiltonian of the system. Suppose that the Hilbert space is n-dimensional (the case $n=+\infty$ is not excluded); then the wave function defines a closed curve $\mathcal{C}$ in the complex projective space $\mathbb{C}P^{n-1}$. If one redefines the wave function such that $\widetilde{\psi}(T)=\widetilde{\psi}(0)$ then

$$\psi(t) = e^{-\imath\hbar^{-1}\int_0^t \langle \widetilde{\psi}(t')|H(t')|\widetilde{\psi}(t')\rangle \mathrm{d}t' - \int_0^t \langle \widetilde{\psi}(t')|\partial_{t'}|\widetilde{\psi}(t')\rangle \mathrm{d}t'} \widetilde{\psi}(t). \tag{3}$$

The extra phase in addition to the dynamical phase is called the Aharonov-Anandan phase (or nonadiabatic Berry phase). We can eliminate the dynamical phase by a gauge transformation; then the Aharonov-Anandan phase appears as the horizontal lift of $\mathcal{C}$ in the principal bundle with base space $\mathbb{C}P^{n-1}$, the structure group U(1) and with the $(2n-1)$-dimensional sphere S^{2n-1} as total space. The Berry-Simon model and the Aharonov-Anandan model are related by the universal classifying theorem of principal bundles;[4,5] more precisely, the Aharonov-Anandan principal bundle is a universal bundle for the Berry-Simon principal bundle.

The two geometric phases described above are called Abelian because they are related to the Abelian group U(1). In 1984, Wilczek and Zee[6] produced a non-Abelian Berry phase phenomenom in the context of the adiabatic approximation. Let $E_a(\vec{R}(t))$ be an M-fold degenerate instantaneous eigenvalue isolated from the rest of the spectrum and $\{|a,i,\vec{R}(t)\rangle\}_{i=1,\ldots,M}$ be an orthonormal basis for the associated eigensubspace. Suppose that the initial state is $\psi(0)=|a,i,\vec{R}(0)\rangle$; then the wave function is

$$\psi(t) = \sum_{j=1}^{M} e^{-\imath\hbar^{-1}\int_0^t E_a(\vec{R}(t'))\mathrm{d}t'} [\mathbb{T}e^{-\int_0^t A(\vec{R}(t'))}]_{ji} |a,j,\vec{R}(t)\rangle, \tag{4}$$

where the matricial 1-form A has the elements $A_{ij}=\langle a,i,\vec{R}|\mathrm{d}_{\mathcal{M}}|a,j,\vec{R}\rangle$ and $\mathbb{T}$ is the time-ordering operator. By elimination of the dynamical phase, this expression becomes a horizontal lift of the curve $\mathcal{C}$ described by $t \mapsto \vec{R}(t)$ into a principal bundle with base space $\mathcal{M}$ and structure group U(M). If $\mathcal{C}$ is closed then $\mathbb{P}e^{-\oint_{\mathcal{C}} A(\vec{R})} \in U(M)$ is the holonomy of the horizontal lift.

In 1994, Bohm and Mostafazadeh[7] constructed a non-Abelian Aharonov-Anandan phase as the universal bundle of the preceding one. Let $P(t)$ be an M-fold degenerate projector such that $P(T)=P(0)$. $t \mapsto P(t)$ defines a closed curve $\mathcal{C}$ in the Grassmanian manifold $G_M(\mathbb{C}^n)$. If one chooses a local section of the bundle $[V_M(\mathbb{C}^n), G_M(\mathbb{C}^n), \mathrm{U}(M), \pi_{V_M(\mathbb{C}^n)}]$ [where $V_M(\mathbb{C}^n)$ is the Stiefel manifold] $s(P)=\{\phi_1(P),\ldots,\phi_M(P)\}$, then the evolution of the wave function for which $\psi(0)=\phi_i(0)$ is given by

$$\psi(t) = \sum_{j,k=1}^{M} [\mathbb{T}e^{-\imath\hbar^{-1}\int_0^t E(t')\mathrm{d}t'}]_{jk} [\mathbb{T}e^{-\int_0^t A(t')}]_{ki} \phi_j(t), \tag{5}$$

where $A_{ij}=\langle \phi_i|\mathrm{d}_{G_M(\mathbb{C}^n)}|\phi_j\rangle$, $E_{ij}(t)=\langle \phi_i(t)|H(t)|\phi_j(t)\rangle$, and where we suppose that $\forall t$ $[\int_0^t E(t')\mathrm{d}t', \int_0^t A(t')]=0$. By eliminating the dynamical phase, we obtain a horizontal lift in the universal bundle.

The gauge potentials defined in the four previous cases have the particular form $A=U^\dagger \mathrm{d}U$ where U is a $M\times n$ matrix († denotes the transconjugation). A connection with such a gauge potential is called a Stiefel connection. It is in fact the form of the universal connection in the universal bundle, a form which is unique, since Narasimhan and Ramaman[8] proved that all universal connections can be written in this form. Then the above four examples of geometric phase each have an explicit Stiefel structure.

In the previous example, the dynamical phase commutes with the geometric phase, but this is not the case in general. More usually we have $\mathbb{T}e^{-\imath\hbar^{-1}\int_0^t E(t')\mathrm{d}t' - \int_0^t A(t')} \neq \mathbb{T}e^{-\imath\hbar^{-1}\int_0^t E(t')\mathrm{d}t'}\mathbb{T}e^{-\int_0^t A(t')}$ and it is then impossible to eliminate the dynamical phase and to obtain a simple geometric structure which reduces the phenomenom to a horizontal lift in a principal bundle.

Sardanashvily introduced[9,10] a model based on a vector composite bundle, so as to obtain a geometric structure which can describe both the dynamical and the geometric phases when they do not commute. His formulation defines the covariant derivative

$$\nabla_h \psi = (\partial_t + \mathcal{A}_\mu(h(t),t)(\partial_t h^\mu(t)) + \imath\hbar^{-1}\mathcal{H}(h(t),t))\psi, \tag{6}$$

where h is a map from $\mathbb{R}$ to $\mathcal{M}$ (the manifold of classical parameters), and $\mathcal{A}$ and $\mathcal{H}$ are bounded operators. This leads to the result that if a section ψ is an integral section of the connection (i.e., $\nabla_h\psi=0$) then

$$\psi(t) = \mathbb{T}e^{-\int_0^t(\mathcal{A}_\mu\partial_{t'}h^\mu+\imath\hbar^{-1}\mathcal{H})\mathrm{d}t'}\psi(0). \tag{7}$$

Sardanashvily finally claims that if one can think of the equation $\nabla_h\psi=0$ as being the Schrödinger equation of a quantum system depending on the parameter $h(t)$, then $\mathcal{A}$ generates the Berry phase and $\mathcal{H}$ generates the dynamical phase. We now explain why we are not in agreement with this claim. First his whole analysis is made in the framework of a vector bundle. This is basically not incorrect; however the Berry phase is usually described in the framework of a principal bundle and its associated vector bundle, giving a principal structure which is more rich. Also, the definition of the covariant derivative used by Saradanashvily is only indirectly justified by noting that it gives a correct final result. More precisely, Sardanashvily does not explicitly define $\mathcal{A}$ and $\mathcal{H}$, whereas the adiabatic potential and the dynamical phase have well-defined matricial expressions. Finally, Eq. (7) is not the expression of a dynamical phase added to a geometric phase, because the expression of Sardanashvily is of the form $\psi(t)=U_{\mathrm{Sar}}(t)\psi(\mathbf{0})$, whereas the expression of a parallel transport with geometric phase is of the form $\psi(t)=U_{\mathrm{phase}}(t)\tilde{\psi}(\mathbf{t})$ where $\tilde{\psi}(\mathbf{t})$ is a known function or a known basis set dependent on the time (in the case of the Berry phase it is the instantaneous eigenvector basis set). This known function (or basis set) defines the local section used to describe the horizontal lift. Then U_{Sar} is not a non-Abelian phase but is the evolution operator. Consequently we have $\imath\hbar\mathcal{A}_\mu\partial_t h^\mu+H=H$, where H is the Hamiltonian of the system, which is split into two parts $\mathcal{A}$ and $\mathcal{H}$. Since Sardanashvily does not define $\mathcal{A}$, this spliting is totally arbitrary. Moreover, if A is the generator of the Berry phase and E is the generator of the dynamical phase, we have $\imath\hbar A+E\neq H$. Thus the affirmation of Sardanashvily in Ref. 9, namely "$\mathcal{A}$ is responsible for the Berry phase phenomenon" appears to be unjustified.

In this paper we construct a geometric structure to give a correct description of the transport when the Berry phase does not commute with the dynamical phase. We apply Sardanashvily's idea of using a composite bundle, but we take a principal structure in place of the vector structure. We thus construct a connection consistent with the geometric model of the adiabatic transport and with the bundle formulation of nonrelativistic quantum dynamics. Our approach reveals that the non-commutativity of A and E introduces a modified gauge structure. Similar modifications of the gauge have been used by Attal in Ref. 11 in a treatment of the non-Abelian gerbes connection. (We note that in the literature of fiber bundle theory the words "gerbe" and "sheaf" are used by various authors to name the same mathematical entity.) He introduced a gauge theory with generalized Cartan structure equations $F=\mathrm{d}A+\frac{1}{2}[A,A]+B$ and $G=\mathrm{d}B+[A,B]$, and a Bianchi identity $\mathrm{d}G+[A,G]=[F,B]$ where A is a 1-form, B and F are 2-forms, and G is a 3-form. In the same way, Larsson,[12] in his treatment of the Yang-Baxter equation with non-Abelian gerbes, found a gauge theory with the usual Cartan equation and the usual Bianchi identity, but limited by restrictions concerning the possible gauge transformations. It should be stressed that the modified gauge theory induced by the noncommutativity of A and E in our approach is similar to the modified gauge theories of Attal and Larsson which are induced by noncommutativity in the gerbes.

This paper is organized as follows. Section II introduces the generalized adiabatic theorem with a Berry phase which does not commute with the dynamical phase. Section III is devoted to some remarks about the principal composite bundles constituting the fundamental structure of our model. Section IV presents the geometric model used for the description of quantum adiabatic dynamics. Finally, Sec. V presents an application to a simple quantum dynamical system. Different quantum dynamical aspects of this system are presented in our geometric representation.

A note about some of the notation used here: the symbol "$\simeq$" between two manifolds denotes that the two manifolds are diffeomorphic, the symbol "$\hookrightarrow$" denotes an inclusion between two sets and "$\Omega^n M$" denotes the set of the n-differential forms of the manifold M.

II. GENERALIZED ADIABATIC TRANSPORT

Theorem 1: *Let $U(t,0)$ be an evolution operator of a quantum dynamical system governed by the self-adjoint Hamiltonian $H(t)$. Let $\{E_a(t)\}_a$ and $\{|a,t\rangle\}_a$ be the instantaneous eigenvalues and eigenvectors of $H(t)$. We suppose that there exists a set of indices I such that the projector $P_m(t)=\sum_{a\in I}|a,t\rangle\langle a,t|$ satisfies the adiabatic condition*

$$\mathrm{U}(t,0)P_m(0) = P_m(t)\mathrm{U}(t,0) \tag{8}$$

(to satisfy this assumption, see for example, Nenciu's adiabatic theorem[13]). If at $t=0$ the wave function is $\psi(0)=|a,0\rangle$ (with $a\in I$) then at time t we have

$$\psi(t) = \sum_{b\in I} U_{ba}(t)|b,t\rangle \tag{9}$$

with the matrix

$$U(t) = \mathbb{T}e^{-\imath\hbar^{-1}\int_0^t E(t')\mathrm{d}t' - \int_0^t A(t')}. \tag{10}$$

Here $A_{ab}(t)=\langle a,t|\partial_t|b,t\rangle\mathrm{d}t$ and we also have $\forall a,b\in I$ $E_{ab}(t)=E_a(t)\delta_{a,b}$.

Proof: The condition (8) states that the evolution is inside Ran P_m so that for all t the wave function can be expanded on the basis set $(|a,t\rangle)_{a\in I}$. This justifies the use of Eq. (9), in which U is a unitary matrix (the unitarity of U results from the normalization of the wave function). The use of Eq. (9) in the Schrödinger equation leads to the result

$$\sum_b \imath\hbar\dot{U}_{ba}(t)|b,t\rangle + \sum_b \imath\hbar U_{ba}(t)\partial_t|b,t\rangle = \sum_b U_{ba}(t)E_b(t)|b,t\rangle. \tag{11}$$

By projecting this expression on $\langle c,t|$ we obtain

$$\imath\hbar\dot{U}_{ca} + \sum_b \imath\hbar U_{ba}\langle c,t|\partial_t|b,t\rangle = U_{ca}E_c \tag{12}$$

which leads to the result

$$(\dot{U}U^{-1})_{cd} = \sum_a \dot{U}_{ca}U^{-1}_{ad} \tag{13}$$

$$= -\sum_a \imath\hbar^{-1}E_c U_{ca}U^{-1}_{ad} - \sum_{a,b}\langle c,t|\partial_t|b,t\rangle U_{ba}U^{-1}_{ad} \tag{14}$$

$$= -\imath\hbar^{-1}E_c(UU^{-1})_{cd} - \sum_b \langle c,t|\partial_t|b,t\rangle(UU^{-1})_{bd} \tag{15}$$

$$= -\imath\hbar^{-1}E_c\delta_{cd} - \langle c,t|\partial_t|d,t\rangle. \tag{16}$$

■

This expression manifestly displays a matrix dynamical phase and a matrix geometric phase, in other words a non-Abelian dynamical phase and a non-Abelian geometric phase. In general $[E,A]\neq 0$. If the Hamiltonian is time-dependent with respect to some classical control parameters $\vec{R}$ which describe a $\mathcal{C}^\infty$-differentiable manifold $\mathcal{M}$, then we can rewrite the non-Abelian phase (10). For a dynamics $H(\vec{R}(t))$ described by a path $\mathcal{C}$ in $\mathcal{M}$ we can set

$$U(\mathcal{C}) = \mathbb{T}e^{-\imath\hbar^{-1}\int_0^t E(\vec{R}(t'))\mathrm{d}t' - \oint_{\mathcal{C}} A(\vec{R})} \tag{17}$$

with $A(\vec{R})_{ab}=\langle a,\vec{R}|\mathrm{d}_{\mathcal{M}}|b,\vec{R}\rangle$, where $d_{\mathcal{M}}$ is the exterior differential of $\mathcal{M}$.

III. PRINCIPAL COMPOSITE BUNDLES

A. Definition of a composite bundle

A composite bundle is defined by five kinds of data, three manifolds (P^+, S, and R) and two surjective maps $\pi_+ : P^+ \to S$ and $\pi_S : S \to R$. We denote the composite bundle by $P^+ \to S \to R$. We assert that a composite bundle is a principal composite bundle if $S \to R$ is a locally trivial fiber bundle with as typical fiber a manifold $M:(S,R,M,\pi_S)$, and if $P^+ \to S$ is a principal bundle with as structure group a Lie group $G:(P^+,S,G,\pi_+)$. $\forall y \in R$, we have $\pi_S^{-1}(y) \simeq M$ and we denote by χ_y^S the associated fiber diffeomorphism. We suppose that S has a structure of a cell complex; then $\pi_S^{-1}(y)$ and M are cell complexes. The theorem of universal classification of principal bundles (see Refs. 4 and 5) is used to define the universal bundle (U,B,G,π_U) where B is the classifying space of M, ϱ_U the universal map from M to B, and $\varrho_U \circ \chi_y^S$ the universal map from $\pi_S^{-1}(y)$ to B. We finally define the principal bundle (P,M,G,π_P) such that the following diagram commutes:

$$\begin{tikzcd}
P^+ \arrow[d, "\pi_+"'] & \chi_y^{S*}P \arrow[l, hook', "\hookleftarrow"'] \arrow[d, "\pi_{P_y}"] & P \arrow[l, "\chi_y^{S*}"'] \arrow[d, "\pi_P"] & U \arrow[l, "\varrho_U^*"'] \arrow[d, "\pi_U"] \\
S & \pi_S^{-1}(y) \arrow[l, hook', "\hookleftarrow"'] \arrow[r, "\chi_y^S"] & M \arrow[r, "\varrho_U"] & B
\end{tikzcd}$$

where $P=\varrho_U^* U$ and $\chi_y^{S^*}P=(\varrho_U \circ \chi_y^S)^* U$. We know that U is independent of $\chi_y^{S^*}P$ and thus independent of y. Moreover, since χ_y^S is a diffeomorphism then $\chi_y^{S^*}$ is a principal bundle isomorphism, so that $\chi_y^{S^*}P=\pi_+^{-1}(\pi_S^{-1}(y))$ and $\pi_{P_y}=\pi_{+|\chi_y^{S^*}P}$. Let U^i be an open local chart on M. We consider the local trivialization of (P,M,G,π_P) over U^i, $\phi_P^i : U^i \times G \to \pi_P^{-1}(U^i)$; then the local trivialization of $(\chi_y^{S^*}P,\pi_S^{-1}(y),G,\pi_{P_y})$ is $\phi_P^i[y]=\chi_y^{S^*}\phi_P^i : \chi_y^{S^{-1}}(U^i)\times G \to \pi_+^{-1}(\chi_y^{S^{-1}}(U^i))$. Let V^j be an open local chart of R, ϕ_S^j be the local trivialization of (S,R,M,π_S) over V^j, and ϕ_+^j be the local trivialization of (P^+,S,G,π_+) over $\pi_S^{-1}(V^j)$. It is clear that the local trivializations are related by

$$\phi_+^j : \begin{array}{l} \pi_S^{-1}(V^j)\times G \to \pi_+^{-1}(\pi_S^{-1}(V^j)) \\ \qquad (s,g) \mapsto \phi_P^i[\mathrm{Pr}_1\, \phi_S^{j^{-1}}(s)](\mathrm{Pr}_2\, \phi_S^{j^{-1}}(s),g), \end{array}$$

where we have supposed that $\mathrm{Pr}_2\, \phi_S^{j^{-1}}(s) \in U^i$. Pr_1 and Pr_2 are the canonical projections of $R \times M$ over R and M. We call (P,M,G,π_P) the structure bundle of the composite bundle.

Finally one can define a principal bundle related to the principal composite bundle. Consider the map

$$\phi_{++}^{ij} : \begin{array}{l} U^i \times V^j \times G \to P^+ \\ \qquad (x,y,g) \mapsto \phi_+^j(\phi_S^j(y,x),g) = \phi_P^i[y](x,g). \end{array}$$

This map is a local trivialization of a principal bundle $(P^+, M\times R, G, \pi_{++})$ with $\pi_{++}=\phi_S^{-1}\circ \pi_+$. It is called the total bundle of the principal composite bundle.

Let $x \in U^i$ be a fixed point of M. Define the map

$$\phi_Q^j[x] : \begin{array}{l} V^j \times G \to P^+ \\ \qquad (y,g) \mapsto \phi_{++}^{ij}(x,y,g). \end{array}$$

If we consider this map as a local trivialization, it defines a principal bundle (Q_x,R,G,π_{G_x}) with $\pi_{Q_x}=\mathrm{Pr}_1\circ \phi_Q^j[x]^{-1}$. We call it the transversal bundle of the principal composite bundle. The situation is schematically summarized in Fig. 1.

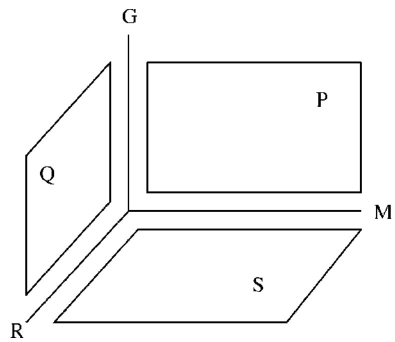

FIG. 1. Scheme of a composite bundle. P^{+} is the three-dimensional space delimited by the three planes.

B. Connection on a principal composite bundle

Consider a principal composite bundle $P^{+}\to S\to R$. (P^{+},S,G,π_{+}) and $(P^{+},M\times R,G,\pi_{++})$ have the structures of principal bundles. We can then define a common connection for these bundles. Let $\omega\in\Omega^{1}(P^{+},\mathfrak{g})$ be the connection 1-form ($\mathfrak{g}$ denotes the Lie algebra of the group G). Let $\sigma^{ij}_{M\times R}\in\Gamma(U^{i}\times V^{i},P^{+})$ be a local section of the principal bundle $(P^{+},M\times R,G,\pi_{++})$. The gauge potential of this bundle is by definition $A^{ij}_{M\times R}=\omega\circ\sigma^{ij}_{M\times R_{*}}\in\Omega^{1}(M\times R,\mathfrak{g})$. Let $\sigma^{j}_{S}\in\Gamma(\pi^{-1}(V^{j}),P^{+})$ be a local section of the bundle (P^{+},S,G,π_{+}). In order to simplify the passage from one bundle to another one, the two sections are chosen to be compatible, i.e., $\forall s\in\pi_{S}^{-1}(V^{j})$, $\sigma^{j}_{S}(s)=\sigma^{ij}_{M\times R}(\phi^{j^{-1}}_{S}(s))$ [we suppose that $\mathrm{Pr}_{1}\,\phi^{j^{-1}}_{S}(s)\in U^{i}$], and $\forall(x,y)\in U^{i}\times V^{j}$, $\sigma^{ij}_{M\times R}(x,y)=\sigma^{j}_{S}(\phi^{j}_{S}(x,y))$. Using these relations the gauge potential of the bundle (P^{+},S,G,π_{+}) is $A^{j}_{S}=\sigma^{j^{*}}_{S}\omega=\phi^{j^{-1^{*}}}_{S}\sigma^{ij}_{M\times R}{}^{*}\omega=\phi^{j^{-1^{*}}}_{S}A^{ij}_{M\times R}\in\Omega^{1}(S,\mathfrak{g})$.

$\chi^{S^{*}}_{y}:P\to\pi_{+}^{-1}(\pi_{S}^{-1}(y))$ is a diffeomorphism, then [remark about the notation: the first star $\chi^{S^{*}}_{y}$ denotes the map induced by χ^{S}_{y} in the principal bundles over $\pi_{S}^{-1}(y)$ and M, the second star $\chi^{S^{**}}_{y}$ denotes the map induced by $\chi^{S^{*}}_{y}$ in the cotangent bundles of P and $\chi^{S^{*}}_{y}P$] $\chi^{S^{**}}_{y}:\Omega^{*}(\pi_{+}^{-1}(\pi_{S}^{-1}(y)))\to\Omega^{*}P$. Let $i_{y}:\pi_{+}^{-1}(\pi_{S}^{-1}(y))\hookrightarrow P^{+}$ be the canonical injection. We define a family of connections of (P,M,G,π_{P}) by $\omega_{y}=\chi^{S^{**}}_{y}i^{*}_{y}\omega\in\Omega^{1}(P,\mathfrak{g})$, for $y\in R$. Let $\sigma^{i}_{M}\in\Gamma(U^{i},P)$ be a local section which is supposed to be compatible with the section of P^{+}: $\forall x\in M$, $\sigma^{i}_{M}(x)=\chi^{S^{*-1}}_{y}\sigma^{ij}_{M\times R}(x,y)$ (this section depends on y). The gauge potential is $A^{i}_{y}(x)=\sigma^{i}_{M}{}^{*}\omega_{y}=\sigma^{ij}_{M\times R}{}^{*}\chi^{S^{*-1^{*}}}_{y}\chi^{S^{**}}_{y}i^{*}_{y}\omega=j^{*}_{y}A^{ij}_{M\times R}(x,y)\in\Omega^{1}(M,\mathfrak{g})$, where $j_{y}:\begin{smallmatrix}M\to M\times R\\ x\mapsto(x,y)\end{smallmatrix}$.

Finally, let $i_{x}:\pi_{++}^{-1}(x,R)\hookrightarrow P^{+}$ be the canonical injection. $\omega_{x}=i^{*}_{x}\omega$ is a connection of the transversal bundle $(Q_{x},R,G,\pi_{Q_{x}})$. Let $j_{x}:\begin{smallmatrix}R\to M\times R\\ y\mapsto(x,y)\end{smallmatrix}$, so that $A_{x}=j^{*}_{x}A_{M\times R}\in\Omega^{1}(R,\mathfrak{g})$ is the gauge potential of this bundle.

C. Horizontal lift in a principal composite bundle

In the theory of principal bundles, the notion of a horizontal lift of a curve of the base space is fundamental. In a principal composite bundle $P^{+}\to S\to R$, there exists a natural generalization of this notion, but the horizontal lift concerns a section of the base bundle (S,R,M,π_{S}). Let $h\in\Gamma(\ell_{R},S)$ be a section, where ℓ_{R} is a curve in R. h defines a curve $\mathcal{C}$ in $M\times R$ parametrized by $y\in R$ with the function $\phi^{j^{-1}}_{S}(h(y))$ (to simplify the discussion we suppose that $\ell_{R}\subset V^{j}$). Let $i_{h}:h(\ell_{R})\hookrightarrow S$ be the canonical injection. We consider the local section $\sigma^{h}\in\Gamma(\mathcal{C},P^{+})$ of the bundle $(P^{+},M\times R,G,\pi_{++})$ defined by $\sigma^{h}=\sigma^{j}_{S}\circ\phi^{j}_{S}$, $\forall y\in R$, $\sigma^{h}(\phi^{j^{-1}}_{S}(h(y)))=\sigma^{j}_{S}(h(y))$. Then the horizontal lift of h is defined as the usual horizontal lift of $\mathcal{C}$ in the total bundle of the composite bundle,

$$g_{h}=\mathbb{P}e^{-\int_{\mathcal{C}}A^{\sigma^{h}}_{M\times R}}=\mathbb{P}e^{-\int_{h(\ell_{R})}i^{*}_{h}A^{j}_{S}}=\mathbb{P}e^{-\int_{\ell_{R}}h^{*}A^{j}_{S}}\tag{18}$$

IV. GEOMETRIC STRUCTURES OF GENERALIZED ADIABATIC TRANSPORT

Consider again the generalized adiabatic transport characterized by the non-Abelian phase (17) in the framework of our quantum mechanical study. The quantum dynamical system is described by a self-adjoint Hamiltonian $t \mapsto H(\vec{R}(t), t)$ in a separable Hilbert space $\mathcal{H}$ where $\vec{R}$ is a set of classical parameters which evolve adiabatically with respect to the quantum proper time. These parameters form a $\mathcal{C}^\infty$-differential manifold $\mathcal{M}$. $t \mapsto \vec{R}(t)$ represents a particular dynamics described by a curve $\mathcal{C}$ in $\mathcal{M}$. To have a more general description, we assume that the dynamics possesses a part which changes rapidly and which cannot be modeled by a classical parameter. The Hamiltonian then has an explicit dependence on t besides having the adiabatic parameters $\vec{R}(t)$.

To control the physical processes it is necessary to model numerous different dynamics, without fixing any particular path in the classical parameter manifold and without fixing the duration of the evolution. Hence we must consider the generic Hamiltonian

$$H: \begin{array}{l} \mathcal{M} \times \mathbb{R} \to \mathcal{L}(\mathcal{H}) \\ (\vec{R}, t) \mapsto H(\vec{R}, t) \end{array} \tag{19}$$

To study this dynamical system, we should separate the dynamical and the geometric contributions to the dynamics. To do this we fix $t \in \mathbb{R}$ in a first step and obtain a purely adiabatic (geometric) evolution $\vec{R} \mapsto H(\vec{R}, t)$. Next we fix $\vec{R} \in \mathcal{M}$ and obtain a purely quantum dynamical evolution $t \mapsto H(\vec{R}, t)$. These two steps are analyzed successively in the next sections A and B.

A. The fiber bundle of the geometry

Let $t_0 \in \mathbb{R}$ be fixed. $\vec{R} \mapsto H(\vec{R}, t_0)$ is the Hamiltonian of an adiabatic system. We suppose that $\{E_a(\vec{R}, t_0)\}_{a\in I}$ are M point eigenvalues of $H(\vec{R}, t_0)$ which form a group which is isolated from the rest of the H spectrum, and we denote by $\{|a, \vec{R}, t_0\rangle\}_{a\in I}$ the corresponding eigenvectors [the case of a globally degenerate eigenvalue is not excluded, but in this case $\exists a, b \in I$ such that $\forall \vec{R} E_a(\vec{R}, t_0) = E_b(\vec{R}, t_0)$ where $(|a, \vec{R}, t_0\rangle, |b, \vec{R}, t_0\rangle)$ is an orthonormal basis of the eigensubspace]. The case of an isolated degeneracy (an eigenvalue crossing) for which $\exists \vec{R}$ such that $E_a(\vec{R}, t_0) = E_b(\vec{R}, t_0)$ can also be included. The works of Berry,[1] Simon,[2] Wilczek and Zee[6] assert that the adiabatic evolution is described mathematically by using a principal bundle $(P, \mathcal{M}, U(M), \pi_P)$ where the connection is represented by the gauge potential $A \in \Omega^1(\mathcal{M}, \mathfrak{u}(M))$,

$$A_{ab}(\vec{R}, t_0) = \langle a, \vec{R}, t_0 | \mathrm{d}_{\mathcal{M}} | b, \vec{R}, t_0\rangle. \tag{20}$$

Here $\mathrm{d}_{\mathcal{M}}$ is the exterior differential of $\mathcal{M}$. This expression is associated with the section of the associated vector bundle $\vec{R} \mapsto (|a, \vec{R}, t_0\rangle)_{a\in I}$.

By introducing the eigenvector-matrix $T(\vec{R}, t_0) \in \mathcal{M}_{\dim \mathcal{H} \times M}(\mathbb{C})$, we can write $A = T^\dagger \mathrm{d}T$, giving a Stiefel connection in agreement with the Narasimhan-Ramaman theorem (see Refs. 14 and 8). It is evident that after a change of section $(\forall a\ |a, \vec{R}, t_0\rangle \rightsquigarrow g(\vec{R})|a, \vec{R}, t_0\rangle$ with $g(\vec{R}) \in U(M))$ the gauge potential satisfies the usual relation

$$\widetilde{A}(\vec{R}, t_0) = g(\vec{R})^{-1} A(\vec{R}, t_0) g(\vec{R}) + g(\vec{R})^{-1} \mathrm{d}_{\mathcal{M}} g(\vec{R}). \tag{21}$$

Note that a family of connections $\{A(\vec{R}, t) \in \Omega^1(\mathcal{M}, \mathfrak{u}(M))\}_{t\in\mathbb{R}}$ is generated if t is continuously modified. If $\mathcal{C}$ is a closed curve in $\mathcal{M}$, then its horizontal lift is characterized by a Wilson loop,

$$W(\mathcal{C}, t_0) = \mathbb{P} e^{-\oint_{\mathcal{C}} A_\mu(\vec{R}, t_0) \mathrm{d}R^\mu}. \tag{22}$$

If the adiabatic bundle is not trivial then $W(\mathcal{C}, t_0) \neq 1$ is the holonomy of the horizontal lift which is called the Berry phase.

B. The fiber bundle of the dynamics

Consider now a fixed point $\vec{R}_0 \in \mathcal{M}$. $t \mapsto H(\vec{R}_0, t)$ is the Hamiltonian describing the quantum dynamics in a static environment which is characterized by the fixed parameters $\vec{R}_0$. Let $t_0, t_1 \in \mathbb{R}$; the evolution of the system between t_0 and t_1 which is described by the evolution operator $U(\vec{R}_0, t_0, t_1)P_m(t_0) \in \mathcal{U}(\mathrm{Ran}\, P_m) \simeq U(M)$ which is assumed to satisfy the adiabatic condition (8), while also being the solution of the Schrödinger equation

$$\imath\hbar \frac{\partial}{\partial t} U(\vec{R}_0, t_0, t) = H(\vec{R}_0, t) U(\vec{R}_0, t_0, t) \tag{23}$$

with $U(\vec{R}_0, t_0, t_0) = 1$. It is well known that the solution of this equation is

$$U(\vec{R}_0, t_0, t_1) = \mathbb{T} e^{-\imath\hbar^{-1} \int_{t_0}^{t_1} H(\vec{R}_0, t) \mathrm{d}t}. \tag{24}$$

Expressions (22) and (24) are very similar, and an interpretation of the quantum dynamics as a parallel transport has been developed by Asorey *et al.*[15] in the general framework of infinite-dimensional Hilbert space and by Iliev[16] in the context of a general fiber bundle model of non-relativistic quantum mechanics. In the context of the adiabatic condition (8) the evolution is condensed into an M-dimensional space, leading to a description which involves a principal bundle $(Q_{\vec{R}_0}, \mathbb{R}, U(M), \pi_{Q_{\vec{R}_0}})$ and its associated vector bundle $(E_{\vec{R}_0}, \mathbb{R}, \mathbb{C}^M, \pi_{E_{\vec{R}_0}})$, with a state appearing as a section of the vector bundle $\psi \in \Gamma(\mathbb{R}, E_{\vec{R}_0})$.

Suppose that $\psi \in \Gamma(\mathbb{R}, E_{\vec{R}_0})$ is a solution of the Schrödinger equation. Let $\widetilde{\psi}(t) = U(t)\psi(t)$ be a change of section such that

$$\imath\hbar \frac{\partial}{\partial t} \widetilde{\psi}(t) = \widetilde{H}(\vec{R}_0, t) \widetilde{\psi}(t). \tag{25}$$

Inserting $\widetilde{\psi}(t) = U(t)\psi$ into Eq. (25) leads to the result

$$\imath\hbar \frac{\partial}{\partial t} \psi(t) = (U(t)^{-1} \widetilde{H}(\vec{R}_0, t) U(t) - \imath\hbar U(t)^{-1} \dot{U}(t)) \psi \tag{26}$$

so that

$$\imath\hbar^{-1} \widetilde{H}(\vec{R}_0, t) = U(t) \imath\hbar^{-1} H(\vec{R}_0, t) U(t)^{-1} + \dot{U}(t) U(t)^{-1} \tag{27}$$

and, finally,

$$\imath\hbar^{-1} \widetilde{H}(\vec{R}_0, t) \mathrm{d}t = U(t) \imath\hbar^{-1} H(\vec{R}_0, t) \mathrm{d}t U(t)^{-1} + (\mathrm{d}_t U(t)) U(t)^{-1}, \tag{28}$$

where d_t is the exterior differential of $\mathbb{R}$: $\mathrm{d}_t f(t) = (\partial f / \partial t)\mathrm{d}t$. Equation (28) is the familiar formula of gauge transformation theory, and $\imath\hbar^{-1} H(\vec{R}_0, t)\mathrm{d}t \in \Omega^1(\mathbb{R}, \mathfrak{u}(M))$ is the gauge potential of $(Q_{\vec{R}_0}, \mathbb{R}, U(M), \pi_{Q_{\vec{R}_0}})$. $\psi \in \Gamma(\mathbb{R}, E_{\vec{R}_0})$ is horizontal if the covariant differential $D\psi = \mathrm{d}_t \psi + \imath\hbar^{-1} H(\vec{R}_0, t)\psi\, \mathrm{d}t$ is zero, i.e., if ψ obeys the Schrödinger equation. A horizontal lift of the curve $[t_0, t_1] \subset \mathbb{R}$ is well characterized by the expression (24).

Let $U_{\mathrm{dyn}}(M)$ be the set of maps from $\mathbb{R}$ to $U(M)$ which satisfies the Schrödinger-von Neumann equation [we note that this equation is the analogue for unitary operators of the equation associated to the Hermitian dynamical invariants (see Ref. 17), which are used by Mostafazadeh in Ref. 18 to define the non-Abelian nonadiabatic geometric phases]

$$\imath\hbar \dot{U}(t) = [U(t), H(\vec{R}_0, t)]. \tag{29}$$

In the framework of the adiabatic condition (8), the previous gauge potential is not used, because (24) is not the expression for the dynamical phase factor appearing in Ref. 19. Let $\{E_a(\vec{R}_0, t)\}_{a \in I}$

be the M isolated eigenvalues of $H(\vec{R}_0,t)$ and $\{|a,\vec{R}_0,t\rangle\}_{a\in I}$ be the corresponding eigenvectors. Let $E(\vec{R}_0,t)$ be the matrix such that $E(\vec{R}_0,t)_{ab}=E_a(\vec{R}_0,t)\delta_{ab}$. It is clear that $E(\vec{R}_0,t)_{ab}=\langle a,\vec{R}_0,t|H(\vec{R}_0,t)|b,\vec{R}_0,t\rangle$. A gauge transformation: $\forall a\in I$, $|\alpha,\vec{R}_0,t\rangle=U(t)|a,\vec{R}_0,t\rangle$, leads to

$$\tilde{E}_{\alpha\beta}(\vec{R}_0,t)=\langle\alpha,\vec{R}_0,t|H(\vec{R}_0,t)|\beta,\vec{R}_0,t\rangle \tag{30}$$

$$=\langle a,\vec{R}_0,t|U(t)^{-1}H(\vec{R}_0,t)U(t)|b,\vec{R}_0,t\rangle \tag{31}$$

$$=\langle a,\vec{R}_0,t|U(t)^{-1}U(t)H(\vec{R}_0,t)|b,\vec{R}_0,t\rangle+\langle a,\vec{R}_0,t|U(t)^{-1}[H(\vec{R}_0,t),U(t)]|b,\vec{R}_0,t\rangle \tag{32}$$

$$=\langle a,\vec{R}_0,t|H(\vec{R}_0,t)|b,\vec{R}_0,t\rangle+\langle a,\vec{R}_0,t|U(t)^{-1}[H(\vec{R}_0,t),U(t)]|b,\vec{R}_0,t\rangle \tag{33}$$

$$=\langle\alpha,\vec{R}_0,t|U(t)H(\vec{R}_0,t)U(t)^{-1}|\beta,\vec{R}_0,t\rangle+\langle\alpha,\vec{R}_0,t|[H(\vec{R}_0,t),U(t)]U(t)^{-1}|\beta,\vec{R}_0,t\rangle \tag{34}$$

$$=(U(t)H(\vec{R}_0,t)U(t)^{-1})_{\alpha\beta}+([H(\vec{R}_0,t),U(t)]U(t)^{-1})_{\alpha\beta}. \tag{35}$$

Thus we have

$$\tilde{E}=UEU^{-1}+[H,U]U^{-1}. \tag{36}$$

E does not satisfy the usual gauge transformation formula. But if we take $U\in U_{\mathrm{dyn}}(M)$ we obtain

$$\imath\hbar^{-1}\tilde{E}(\vec{R}_0,t)\mathrm{d}t=U(t)\imath\hbar^{-1}E(\vec{R}_0,t)\mathrm{d}tU(t)^{-1}+(\mathrm{d}_tU(t))U(t)^{-1} \tag{37}$$

we see that $\imath\hbar^{-1}E(\vec{R}_0,t)\mathrm{d}t\in\Omega^1(\mathbb{R},\mathfrak{u}(M))$ is a gauge potential of the principal bundle $(Q_{\vec{R}_0},\mathbb{R},U(M),\pi_{Q_{\vec{R}_0}})$ but with a restriction of the gauge transformations to the set $U_{\mathrm{dyn}}(M)$ of the sections of $(Q_{\vec{R}_0},\mathbb{R},U(M),\pi_{Q_{\vec{R}_0}})$ which are horizontal for the connection $\imath\hbar^{-1}H(\vec{R}_0,t)\mathrm{d}t$. The horizontal lift of $[t_0,t_1]\subset\mathbb{R}$ is then

$$D(\vec{R}_0,t_0,t_1)=\mathbb{T}e^{-\imath\hbar^{-1}\int_{t_0}^{t_1}E(\vec{R}_0,t)\mathrm{d}t} \tag{38}$$

which is effectively the dynamical term of (17).

The expression for the gauge potential is associated with the section of the associated vector bundle $t\mapsto(|a,\vec{R}_0,t\rangle)_{a\in I}$.

C. The composite bundle of the geodynamics and its connection

The discussion in the two preceding sections A and B suggests that the appropriate entities to give a correct description of the geometric structure of the geodynamical evolution characterized by the expression (17) would be the principal composite bundle $P^+\to S\to\mathbb{R}$ with structure bundle $(P,\mathcal{M},U(M),\pi_P)$, base bundle $(S,\mathbb{R},\mathcal{M},\pi_S)$, transversal bundles $(Q_{\vec{R}},\mathbb{R},U(M),\pi_{Q_{\vec{R}}})$ and total bundle $(P^+,\mathcal{M}\times\mathbb{R},U(M),\pi_{++})$. Note that, following the treatment of Sec. II, the structures of $(P,\mathcal{M},U(M),\pi_P)$, $(Q_{\vec{R}},\mathbb{R},U(M),\pi_{Q_{\vec{R}}})$ and of $(S,\mathbb{R},\mathcal{M},\pi_S)$ completely determine $(P^+,\mathcal{M}\times\mathbb{R},U(M),\pi_{++})$. P and $Q_{\vec{R}}$ have been introduced in the preceding paragraph. By fixing $t_0\in\mathbb{R}$ arbitrarily, the transition functions (to define a principal bundle, there are three equivalent ways, by invoking the local trivializations, by invoking the transition functions or by invoking the fiber diffeomorphisms) $g^{ij}(\vec{R},t_0)\in U(M)$ of P are obtained by setting $g^{ij}_{ab}(\vec{R})=\langle a,\vec{R},t_0,i|b,\vec{R},t_0,j\rangle$ where i and j represent two possible conventions in the matrix representation of the eigenvectors (see the example of Berry phase in Ref. 19). The topology of the bundle S is determined by the map χ_t^S, for which P_t as defined by the transition functions $T(\vec{R},t,i)^\dagger T(\vec{R},t,j)$ is such that $\chi_t^{S^*}P = \chi_t^{S^*}P_{t_0}=P_t$. Clearly, if we consider $|a,\vec{R},t\rangle$ as a section of $\mathcal{M}$ with values in the associated vector bundle of P_t, then we have

$$\chi^S_{t*}|a,\vec{R},t_0\rangle = |a,\vec{R},t\rangle \Leftrightarrow \chi^S_{t*} = U(\vec{R},t_0,t), \tag{39}$$

$U(\vec{R},t_0,t)$ is defined by Eq. (24), and χ^S_{t*} is the map induced by χ^S_t in the sections.

This naturally leads us to take as the gauge potential of $(P_+,\mathcal{M}\times\mathbb{R},U(M),\pi_{++})$ the quantity

$$A_{\mathcal{M}\times\mathbb{R}}(\vec{R},t) = A(\vec{R},t) + \imath\hbar^{-1}E(\vec{R},t)\mathrm{d}t \in \Omega^1(\mathcal{M}\times\mathbb{R},\mathfrak{u}(M)) \tag{40}$$

with A being defined by Eq. (20). Note that we can use this expression because the two local sections used to express the gauge potentials are compatible. Introducing $j_t: {\mathcal{M}\to\mathcal{M}\times\mathbb{R} \atop \vec{R}\mapsto(\vec{R},t)}$ and $j_{\vec{R}}: {\mathbb{R}\to\mathcal{M}\times\mathbb{R} \atop t\mapsto(\vec{R},t)}$ we have $A(\vec{R},t_0)=j^*_{t_0}A_{\mathcal{M}\times\mathbb{R}}(\vec{R},t_0)$ and $\imath\hbar^{-1}E(\vec{R}_0,t)\mathrm{d}t=j^*_{\vec{R}_0}A_{\mathcal{M}\times\mathbb{R}}(\vec{R}_0,t)$.

As a gauge potential is locally defined, and as S is locally diffeomorphic to $\mathcal{M}\times\mathbb{R}$, we can write $A_S(\vec{R},t)=A_\mu(\vec{R},t)\mathrm{d}R^\mu+\imath\hbar^{-1}E(\vec{R},t)\mathrm{d}t\in\Omega^1(S,\mathfrak{u}(M))$. Let $h\in\Gamma([0,t_1],S)$ be a section. h defines a curve $\mathcal{C}$ in $\mathcal{M}\times\mathbb{R}$, where $\mathcal{L}=h([0,t_1])$ is a closed path described in $\mathcal{M}$ by $R^\mu(t)=h^\mu(t)$. Consider the pullback of h,

$$\Omega^* S \to \Omega^*\mathbb{R}$$

$$h^*: \mathrm{d}R^\mu \mapsto \frac{\partial h^\mu}{\partial t}\mathrm{d}t.$$

$$\mathrm{d}t \mapsto \mathrm{d}t$$

Then we have

$$(h^*A_S)(t) = A_\mu(h(t),t)\frac{\partial h^\mu}{\partial t}\mathrm{d}t + \imath\hbar^{-1}E(h(t),t)\mathrm{d}t. \tag{41}$$

Using expression (18), the horizontal lift of h is characterized by [with the notation $h(t)=\vec{R}(t)$]

$$g_h = \mathbb{P}e^{-\int_{\mathcal{C}}A_{\mathcal{M}\times\mathbb{R}}(\vec{R},t)} = \mathbb{T}e^{-\int_0^{t_1}A_\mu(\vec{R}(t),t)[\partial R^\mu(t)/\partial t]\mathrm{d}t - \imath\hbar^{-1}\int_0^{t_1}E(\vec{R}(t),t)\mathrm{d}t}. \tag{42}$$

Suppose now that we do not have a fast evolution in addition to the adiabatic evolution, in such a way that $H(\vec{R},t)$ has no explicit time dependence; then we have

$$g_h = \mathbb{T}e^{-\oint_{\mathcal{L}}A_\mu(\vec{R})\mathrm{d}R^\mu - \imath\hbar^{-1}\int_0^{t_1}E(\vec{R}(t))\mathrm{d}t} \tag{43}$$

which is the expression for the non-Abelian phase in (17).

Note that the connection A_S of $(P^+,S,U(M),\pi_+)$ is restricted to the gauge transformations of the form $U(M)\ni g(\vec{R},t)=g(\vec{R})U(t)$, where $g(\vec{R})$ is a map from $\mathcal{M}$ to $U(M)$ without restrictive conditions and where $U(t)\in U_{\mathrm{dyn}}(M)$. We thus have a principal structure but with a restricted choice of gauges.

It should be stressed that in $Q_{\vec{R}}$ we introduce the local fiberd coordinates (t,γ^j) where (γ^j) is a system of coordinates of $U(M)$. In the same way we introduce the fiberd coordinates of P (R^μ,γ^j) and the fiberd coordinates of P^+ (R^μ,t,γ^j). By calling on the theorem of Ehresmann (see Ref. 19), it is possible to construct a connection 1-form with gauge potential $A_{\mathcal{M}\times\mathbb{R}}$. Let $\sigma\in\Gamma(\mathcal{M}\times\mathbb{R},P^+)$ be the section used to express the gauge potential $A_{\mathcal{M}\times\mathbb{R}}$. $\forall(\vec{R},t,\gamma)\in P^+$, let $g(\vec{R},t,\gamma)$ such that $(\vec{R},t,\gamma)=\sigma(\vec{R},t)g(\vec{R},t,\gamma)$. The connection 1-form of P^+ is

$$\omega_+(\vec{R},t,\gamma) = g(\vec{R},t,\gamma)^{-1}A_\mu(\vec{R},t)g(\vec{R},t,\gamma)\mathrm{d}R^\mu + \imath\hbar^{-1}g(\vec{R},t,\gamma)^{-1}E(\vec{R},t)g(\vec{R},t,\gamma)\mathrm{d}t + g(\vec{R},t,\gamma)^{-1}\frac{\partial}{\partial R^\mu}g(\vec{R},t,\gamma)\mathrm{d}R^\mu + g(\vec{R},t,\gamma)^{-1}\frac{\partial}{\partial t}g(\vec{R},t,\gamma)\mathrm{d}t + g(\vec{R},t,\gamma)^{-1}\frac{\partial}{\partial \gamma^j}g(\vec{R},t,\gamma)\mathrm{d}\gamma^j$$

the connection 1-form of P for a fixed t_0 is

$$\omega_P(\vec{R},\gamma) = g(\vec{R},t_0,\gamma)^{-1}A_\mu(\vec{R},t_0)g(\vec{R},t_0,\gamma)\mathrm{d}R^\mu + g(\vec{R},t_0,\gamma)^{-1}\frac{\partial}{\partial R^\mu}g(\vec{R},t_0,\gamma)\mathrm{d}R^\mu + g(\vec{R},t_0,\gamma)^{-1}\frac{\partial}{\partial \gamma^j}g(\vec{R},t_0,\gamma)\mathrm{d}\gamma^j$$

and the connection 1-form of $Q_{\vec{R}_0}$ for a fixed $\vec{R}_0$ is

$$\omega_{Q_{\vec{R}_0}}(t,\gamma) = \imath\hbar^{-1}g(\vec{R}_0,t,\gamma)^{-1}E(\vec{R}_0,t)g(\vec{R}_0,t,\gamma)\mathrm{d}t + g(\vec{R}_0,t,\gamma)^{-1}\frac{\partial}{\partial t}g(\vec{R}_0,t,\gamma)\mathrm{d}t + g(\vec{R}_0,t,\gamma)^{-1}\frac{\partial}{\partial \gamma^j}g(\vec{R}_0,t,\gamma)\mathrm{d}\gamma^j$$

we can see then that $\omega_{P^+} \neq \omega_P + \omega_{Q_{\vec{R}}}$.

D. A pseudo-Stiefel structure

In the preceding section C we considered a fiber bundle with a restriction concerning the allowed gauge transformations. If we give up this restriction we must deal with the nonstandard equation (36) of gauge tranformation theory, $\widetilde{E} = UEU^{-1} + [H,U]U^{-1}$. In order to find a structure associated with this formula we first consider the bundle $(Q_{\vec{R}_0}, \mathbb{R}, U(M), \pi_{Q_{\vec{R}_0}})$ endowed with the gauge potential $\imath\hbar^{-1}H(\vec{R}_0,t)\mathrm{d}t$. This gauge potential satisfies the correct gauge transformation formula and it is then possible to define a covariant differential $D_{Q_{\vec{R}_0}}$. Let $\psi \in \Gamma(\mathbb{R}, E_{\vec{R}_0})$ be a section of the associated vector bundle. We have

$$D_{Q_{\vec{R}_0}}\psi = \partial_t\psi\,\mathrm{d}t + \imath\hbar^{-1}H\psi\,\mathrm{d}t. \tag{44}$$

Let $U \in \Gamma(\mathbb{R}, Q_{\vec{R}_0})$ be a section from $\mathbb{R}$ to $Q_{\vec{R}_0}$ considered as the space of the operators of $E_{\vec{R}_0}$; then we have

$$D_{Q_{\vec{R}_0}}U = \partial_t U\,\mathrm{d}t + \imath\hbar^{-1}[H,U]\mathrm{d}t. \tag{45}$$

With $D_{Q_{\vec{R}_0}}$ we define a differential in $\mathcal{M}\times\mathbb{R}$,

$$\widetilde{D}\eta(\vec{R},t) = \mathrm{d}_{\mathcal{M}}\eta(\vec{R},t) + D_{Q_{\vec{R}}}\eta(\vec{R},t). \tag{46}$$

We can then define a gauge potential in the style of Stiefel but with the differential $\widetilde{D}$ in place of $\mathrm{d}_{\mathcal{M}\times\mathbb{R}}$. We set

$$A_+ = T^\dagger\widetilde{D}T, \tag{47}$$

where T is the matrix of the eigenvectors of H. Note that $\widetilde{D}^2 \neq 0$ so that the connection is not rigourously a Stiefel one. $|b,\vec{R},t\rangle \in \Gamma(\mathcal{M}\times\mathbb{R}, E_+)$ is a section of the associated vector bundle of P^+, so that

$$\tilde{D}|b,\vec{R},t\rangle = d_{\mathcal{M}}|b,\vec{R},t\rangle + d_t|b,\vec{R},t\rangle + \imath\hbar^{-1}H(\vec{R},t)|b,\vec{R},t\rangle \mathrm{d}t \tag{48}$$

and

$$A_{+ab} = \langle a,\vec{R},t|d_{\mathcal{M}}|b,\vec{R},t\rangle + \langle a,\vec{R},t|\partial_t|b,\vec{R},t\rangle \mathrm{d}t + \imath\hbar^{-1}\langle a,\vec{R},t|H(\vec{R},t)|b,\vec{R},t\rangle \mathrm{d}t \tag{49}$$

so that we have

$$A_+ = A + A_0 + \imath\hbar^{-1}E\mathrm{d}t, \tag{50}$$

where A is the adiabatic gauge potential defined by Eq. (20) and A_0 is the matrix with elements $\langle a,\vec{R},t|\partial_t|b,\vec{R},t\rangle \mathrm{d}t$, namely the expression of a Berry gauge potential for the variable t when $\vec{R}$ is fixed. Considering A_+ as a gauge potential of $(P^+,\mathcal{M}\times\mathbb{R},U(M),\pi_{++})$, the horizontal lift of $h \in \Gamma(\mathbb{R},\mathcal{M})$ is characterized by

$$g_h = \mathbb{T}e^{-\int_0^{t_1}A_\mu(\vec{R}(t),t)[\partial h^\mu(t)/\partial t] - \int_0^{t_1}A_0(\vec{R}(t),t)\mathrm{d}t - \imath\hbar^{-1}\int_0^{t_1}E(\vec{R}(t),t)\mathrm{d}t}. \tag{51}$$

Note that this equation is identical to (10) if one considers the change of variable $t \to (\vec{R}(t),t)$.

Consider a gauge transformation, it is clear that

$$\hat{A}_+ = \hat{A} + \hat{A}_0 + \imath\hbar^{-1}\hat{E}\,\mathrm{d}t \tag{52}$$

$$= UAU^{-1} + (\mathrm{d}_{\mathcal{M}}U)U^{-1} + UA_0U^{-1} + (\mathrm{d}_tU)U^{-1} + U\imath\hbar^{-1}E\,\mathrm{d}t\,U^{-1} + \imath\hbar^{-1}[H,U]U^{-1} \tag{53}$$

$$= UA_+U^{-1} + (\mathrm{d}_{\mathcal{M}}U + \mathrm{d}_tU + \imath\hbar^{-1}[H,U])U^{-1} \tag{54}$$

$$= UA_+U^{-1} + (\tilde{D}U)U^{-1}. \tag{55}$$

A_+ satisfies a gauge transformation formula analogous to the usual one but with the replacement of $\mathrm{d}_{\mathcal{M}\times\mathbb{R}}$ by $\tilde{D}$. The use of the pseudodifferential $\tilde{D}$ modifies the gauge field theory. Let the curvature F_+ be

$$F_+ = \tilde{D}A_+ + A_+\wedge A_+ \tag{56}$$

$$= \mathrm{d}_{\mathcal{M}\times\mathbb{R}}A_+ + A_+\wedge A_+ + \imath\hbar^{-1}[E\,\mathrm{d}t,A_+] \tag{57}$$

$$= \mathrm{d}_{\mathcal{M}\times\mathbb{R}}A_+ + A_+\wedge A_+ + \imath\hbar^{-1}[E\,\mathrm{d}t,A]. \tag{58}$$

Let $B=\imath\hbar^{-1}[E\,\mathrm{d}t,A] \in \Omega^2(\mathcal{M}\times\mathbb{R},\mathfrak{g})$ be the curving. B is the field which characterizes the noncommutativity between the dynamical and the geometric phases. Note that $F_+ - B$ is a standard curvature which satisfies the usual Bianchi identity and the usual Cartan structure equation. Using the standard covariant differential associated with A_+ we obtain the generalized Cartan equations

$$F_+ = \mathrm{d}_{\mathcal{M}\times\mathbb{R}}A_+ + A_+\wedge A_+ + B, \tag{59}$$

$$G = \mathrm{d}_{\mathcal{M}\times\mathbb{R}}B + [A_+,B]. \tag{60}$$

$G \in \Omega^3(\mathcal{M}\times\mathbb{R},\mathfrak{g})$ is called the fake curvature.

The fake-curvature satisfies a pseudo-Bianchi identity.

Property 1: Let G be a fake-curvature defined by generalized Cartan structure equations; then

$$\mathrm{d}_{\mathcal{M}\times\mathbb{R}}G + [G,A_+] = [F_+,B]. \tag{61}$$

Proof:

$$\mathrm{d}G = \mathrm{d}A_+ \wedge B - A_+ \wedge \mathrm{d}B - \mathrm{d}B \wedge A_+ - B \wedge \mathrm{d}A_+, \tag{62}$$

$$[F,B] - [G,A_+] = F_+ \wedge B - B \wedge F_+ - G \wedge A_+ - A_+ \wedge G \tag{63}$$

$$\begin{aligned} &= \mathrm{d}A_+ \wedge B + A_+ \wedge A_+ \wedge B + B \wedge B - B \wedge \mathrm{d}A_+ - B \wedge A_+ \wedge A_+ - B \wedge B - \mathrm{d}B \wedge A_+ - A_+ \wedge B \wedge A_+ \\ &\quad + B \wedge A_+ \wedge A_+ - A_+ \wedge \mathrm{d}B - A_+ \wedge A_+ \wedge B + A_+ \wedge B \wedge A_+ \end{aligned} \tag{64}$$

$$= \mathrm{d}A_+ \wedge B - B \wedge \mathrm{d}A_+ - \mathrm{d}B \wedge A_+ - A_+ \wedge \mathrm{d}B. \tag{65}$$

■

V. ILLUSTRATION: THE EXAMPLE OF A SIMPLE QUANTUM DYNAMICAL SYSTEM

This final section illustrates the formal concepts introduced in the preceding sections by using a concrete physical example taken from atomic physics. We consider a particular three-level atom interacting with two lasers. Before explaining how the model illustrates the formal theory of the preceding sections we give a brief description of three-level systems.

A. Preliminary discussion

We consider a three-level quantum system, described by the Hilbert space $\mathcal{H} = \mathbb{C}^3$. The generic form of a three-level Hamiltonians is

$$H = x^i \lambda_i, \quad i = 0, \ldots, 8, \tag{66}$$

where λ_0 is the identity matrix of $\mathcal{H} = \mathbb{C}^3$ and $\{\lambda_i\}_{i=1,\ldots,8}$ are the Gell-Mann matrices,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\iota & 0 \\ \iota & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\iota \\ 0 & 0 & 0 \\ \iota & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\iota \\ 0 & \iota & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The Gell-Mann matrices can be considered as the generators of the Lie algebra $\mathfrak{su}(3)$. Moreover we introduce the following matrices:

$$\mu_1 = \lambda_3 + \frac{1}{\sqrt{3}}\lambda_8 + \frac{2}{3}\lambda_0, \tag{67}$$

$$\mu_2 = -\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8 + \frac{2}{3}\lambda_0, \tag{68}$$

$$\mu_3 = -\frac{1}{\sqrt{3}}\lambda_8 + \frac{2}{3}\lambda_0. \tag{69}$$

It is clear that $\{\lambda_i, \mu_j\}_{i=1,2,4,5,6,7;j=1,2,3}$ generate the Lie algebra $\mathfrak{u}(3)$. We are interested in particular Hamiltonians of the form

$$H = x^1\lambda_1 + x^2\lambda_2 + x^6\lambda_6 + x^7\lambda_7 + \tilde{x}\mu_2. \tag{70}$$

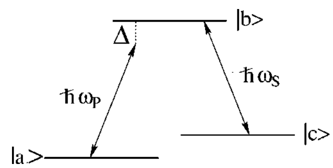

FIG. 2. Scheme of the three-level atom.

Since $\{\lambda_4, \lambda_5, \mu_1, \mu_2\}$ are the generators of the Lie algebra $\mathfrak{u}(2)$ as a subalgebra of $\mathfrak{u}(3)$, then the Hamiltonian (70) is an element of $\mathfrak{u}(3)/\mathfrak{u}(2)$ [this is a vector space quotient; $\mathfrak{u}(2)$ is not an ideal of $\mathfrak{u}(3)$ and so $\mathfrak{u}(3)/\mathfrak{u}(2)$ is a vector space without the Lie algebra structure]. In other words the Hamiltonian (70) characterized by $(x^1, x^2, x^6, x^7, \tilde{x})$ is determined by a point of the manifold $U(3)/U(2)$, and we know (see Ref. 4) that

$$U(3)/U(2) = SU(3)/SU(2) = S^5. \tag{71}$$

Thus, the control parameter space associated with a Hamiltonian of the form (70) can always be chosen as a submanifold of the 5-sphere S^5.

B. A concrete example: a three-level atom interacting with lasers

We consider a three-level atom in the Λ configuration interacting with two lasers, denoted by P (for pump) and S (for Stokes). The three bare states of the atom are labelled by $|a\rangle$, $|b\rangle$, and $|c\rangle$. The control parameters of the system are the amplitudes and the phases of the lasers S and P. We denote by ω_P the frequency of the laser P which is quasiresonant with the transition $|a\rangle \to |b\rangle$, with the detuning Δ. The laser S of frequency ω_S is supposed to be in perfect resonance with the transition $|b\rangle \to |c\rangle$, see Fig. 2.

The dressed Hamiltonian of the system in the rotating wave approximation (RWA) is (see, for example Ref. 20)

$$H = \frac{\hbar}{2}\begin{pmatrix} 0 & We^{\imath\beta} & 0 \\ We^{-\imath\beta} & 2\Delta & Ve^{\imath\alpha} \\ 0 & Ve^{-\imath\alpha} & 0 \end{pmatrix}, \tag{72}$$

where $W = |\langle a|\vec{\mu}\cdot\vec{E}_P|b\rangle|$ and $V = |\langle b|\vec{\mu}\cdot\vec{E}_S|c\rangle|$, $\vec{E}_P$ and $\vec{E}_S$ being the laser amplitudes and $\vec{\mu}$ being the electric dipole moment of the atom. To simplify the notation, we set $\Delta = 1$. The Hamiltonian H is of the form (70) and we can compute the three eigenvalues of H,

$$E_1 = 0, \tag{73}$$

$$E_2 = \frac{\hbar}{2}(1 - \sqrt{1 + V^2 + W^2}), \tag{74}$$

$$E_3 = \frac{\hbar}{2}(1 + \sqrt{1 + V^2 + W^2}). \tag{75}$$

We see that $E_1 = E_2$ if $V = 0$ and $W = 0$, and moreover

$$\inf_{V,W} \mathrm{dist}(\{E_1, E_2(V,W)\}; E_3(V,W)) = \hbar. \tag{76}$$

Let $P_1(W,V,\alpha,\beta)$, $P_2(W,V,\alpha,\beta)$, and $P_3(W,V,\alpha,\beta)$ be the eigenprojectors associated with E_1, E_2, and E_3. It is evident that for all particular dynamics $t \mapsto (W(t), V(t), \alpha(t), \beta(t))$ the Hamiltonian $H(t)$ and the decomposition $\mathrm{Spe}(H(t)) = \sigma_0(t) \cup \sigma_\perp(t)$ satisfy the assumptions of Nenciu's adiabatic theorem (see Ref. 13), where $\sigma_0(t) = \{E_1, E_2(t)\}$ and $\sigma_\perp(t) = \{E_3(t)\}$ and with

$\inf_t \operatorname{dist}(\sigma_0(t),\sigma_\perp(t)) \geqslant \hbar$. In accordance with Nenciu's theorem, we have at the adiabatic limit [this limit is approximately obtained if the variations of $(W(t),V(t))$ are slow with respect to the proper quantum time $\inf_t[\hbar/E_3(t)-E_1]$; for classical parameters such as W or V this hypothesis is consistent]

$$U(t,0)P_m(0) = P_m(t)U(t,0), \tag{77}$$

where $U(t,0)$ is the evolution operator associated with $H(t)$ and $P_m(t)=P_1(t)+P_2(t)$. We can apply the formalism of the previous part with Ran P_m (dim Ran $P_m=2$), for all particular dynamics.

The eigenvectors of H can be chosen as follows for $V\neq 0$ and $W\neq 0$:

$$|1,(\alpha,\beta,W,V)\rangle = \frac{1}{\sqrt{1+\frac{V^2}{W^2}}}\begin{pmatrix} -e^{\iota(\alpha+\beta)}\frac{V}{W} \\ 0 \\ 1 \end{pmatrix}, \tag{78}$$

$$|2,(\alpha,\beta,W,V)\rangle = \frac{1}{\sqrt{1+\frac{W^2}{V^2}+\frac{(1-\sqrt{1+V^2+W^2})^2}{V^2}}}\begin{pmatrix} e^{\iota(\alpha+\beta)}\frac{W}{V} \\ e^{\iota\alpha}\frac{(1-\sqrt{1+V^2+W^2})}{V} \\ 1 \end{pmatrix}, \tag{79}$$

$$|3,(\alpha,\beta,W,V)\rangle = \frac{1}{\sqrt{1+\frac{W^2}{V^2}+\frac{(1+\sqrt{1+V^2+W^2})^2}{V^2}}}\begin{pmatrix} e^{\iota(\alpha+\beta)}\frac{W}{V} \\ e^{\iota\alpha}\frac{(1+\sqrt{1+V^2+W^2})}{V} \\ 1 \end{pmatrix}. \tag{80}$$

Let $r=\sqrt{1+V^2+W^2}$ and (θ,φ) be such that $W=r\sin\varphi\cos\theta$, $V=r\sin\varphi\sin\theta$ and $r\cos\varphi=1$ ($\theta\in\,]0,\pi/2[$ and $\varphi\in\,]0,\pi/2[$). With these variables we can write

$$|1,(\alpha,\beta,\theta,\varphi)\rangle = \begin{pmatrix} -e^{\iota(\alpha+\beta)}\sin\theta \\ 0 \\ \cos\theta \end{pmatrix}, \tag{81}$$

$$|2,(\alpha,\beta,\theta,\varphi)\rangle = \begin{pmatrix} e^{\iota(\alpha+\beta)}\frac{\sin\varphi\cos\theta}{\sqrt{1-\cos\varphi}} \\ e^{\iota\alpha}\sqrt{1-\cos\varphi} \\ \frac{\sin\varphi\sin\theta}{\sqrt{1-\cos\varphi}} \end{pmatrix}. \tag{82}$$

Let $(\alpha,\beta,\gamma,\theta,\varphi)$ be the angles which generate S^5. The submanifold of S^5 defined by

$$0<\varphi<\frac{\pi}{2},$$

$$0<\theta<\frac{\pi}{2},$$

$$\gamma = 0,$$

is the control manifold; in the following we denote it by S^4_+.

C. The composite bundle modelling the quantum dynamical system

We will now apply the theoretical construction introduced in the preceding sections. First we note that the choice of the eigenvectors (81) and (82) is not unique. They can be

$$|i\rangle^{NE} = \begin{pmatrix} e^{\iota(\alpha+\beta)}* \\ e^{\iota\alpha}* \\ * \end{pmatrix}, \tag{83}$$

where $*$ replace the functions of (θ,φ) in the expressions of $|1\rangle$ (81) or $|2\rangle$ (82). By another choice of phase convention we can choose the following eigenvectors:

$$|i\rangle^{NW} = \begin{pmatrix} e^{\iota\alpha}* \\ e^{\iota(\alpha-\beta)}* \\ e^{-\iota\beta}* \end{pmatrix}, \quad |i\rangle^{SE} = \begin{pmatrix} e^{\iota\beta}* \\ * \\ e^{-\iota\alpha}* \end{pmatrix}, \quad |i\rangle^{SW} = \begin{pmatrix} * \\ e^{-\iota\beta}* \\ e^{-\iota(\alpha+\beta)}* \end{pmatrix}. \tag{84}$$

These different conventions are associated with four open local charts of S^4_+, $U^{NE} = \{(\alpha,\beta,\theta,\varphi) \in S^4_+ | \alpha \in]-\pi/2-\epsilon, \pi/2+\epsilon[, \beta \in]-\pi/2-\epsilon, \pi/2+\epsilon[$, $U^{NW} = \{(\alpha,\beta,\theta,\varphi) \in S^4_+ | \alpha \in]-\pi/2-\epsilon, \pi/2+\epsilon[, \beta \in]\pi/2-\epsilon, 3\pi/2+\epsilon[$, $U^{SE} = \{(\alpha,\beta,\theta,\varphi) \in S^4_+ | \alpha \in]\pi/2-\epsilon, 3\pi/2+\epsilon[, \beta \in]-\pi/2-\epsilon, \pi/2+\epsilon[$, and $U^{SW} = \{(\alpha,\beta,\theta,\varphi) \in S^4_+ | \alpha \in]\pi/2-\epsilon, 3\pi/2+\epsilon[, \beta \in]\pi/2-\epsilon, 3\pi/2+\epsilon[$, where ϵ is a small parameter. The set $\{U^i\}_{i=NE,NW,SE,SW}$ is an atlas of S^4_+. We want to construct the principal bundle of the geometric phase. Let $T^i = (|1\rangle^i, |2\rangle^i) \in \mathcal{M}_{3\times 2}(\mathbb{C})$ be the matrix of eigenvectors selected by the adiabatic theorem $(i = NE, NW, SE, SW)$. We set $\vec{R} = (\alpha,\beta,\theta,\varphi) \in S^4_+$,

$$\forall i,j,\ \forall \vec{R} \in U^i \cap U^j, \quad g^{ij}(\vec{R}) = T^i(\vec{R})^\dagger T^j(\vec{R}) \in \mathrm{U}(2). \tag{85}$$

The functions g^{ij} are the transition functions of the principal bundle of the geometry $(P, S^4_+, \mathrm{U}(2), \pi_P)$. More precisely we have

$$g^{NE,NW} = g^{SE,SW} = e^{\iota\beta}, \quad g^{NE,SE} = g^{NW,SW} = e^{\iota\alpha}, \quad g^{NE,SW} = e^{\iota(\alpha+\beta)}. \tag{86}$$

Note that $\forall i,j\ g^{ij} \in \mathrm{U}(1) \subset \mathrm{U}(2)$, because the two eigenvectors are never globally degenerate in $U^i \cap U^j$. These functions define completely the total space P of the principal bundle of the geometry. Indeed let $\sim$ be the equivalence relation on $S^4_+ \times \mathrm{U}(2)$ defined by

$$(x,k) \sim (y,h) \text{ if } x = y \text{ and if } \exists\, i,j \text{ such that } x \in U^i \cap U^j \text{ and } k = hg^{ij}.$$

The total space is the quotient manifold $P = S^4_+ \times \mathrm{U}(2)/\sim$. Let $\pi_\sim : S^4_+ \times \mathrm{U}(2) \to P$ be the projection associated to $\sim$, then π_P is defined by the commutative diagram

$$\begin{tikzcd} S^4_+ \times U(2) \arrow[rr, "\pi_\sim"] \arrow[dr, "\mathrm{Pr}_1"'] & & P \arrow[dl, "\pi_P"] \\ & S^4_+ & \end{tikzcd}$$

The principal bundle of the geometry $(P, S^4_+, \mathrm{U}(2), \pi_P)$ is then completely defined. Moreover it is the structure bundle of the principal composite bundle of the geodynamics. The connection on P is obtained by the gauge potential

$$\forall \vec{R} \in U^i, \quad A^i = T^i(\vec{R})^\dagger \mathrm{d}_{S^4} T^i(\vec{R}) \in \Omega^1(S^4_+, \mathfrak{u}(2)) \tag{87}$$

with $A^j=(g^{ij})^{-1}A^i g^{ij}+(g^{ij})^{-1}\mathrm{d}_{S^4}g^{ij}$ in $U^i \cap U^j$. Let $\{\sigma_i\}_{i=1,2,3}$ be the Pauli matrices [generators of $\mathfrak{su}(2)$] and σ_0 be the identity of $\mathbb{C}^2$, with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The calculus of the gauge potential shows that

$$\begin{aligned} A^{NE} = {} & \iota \frac{\sin\varphi}{\sqrt{1-\cos\varphi}}\sigma_2\, \mathrm{d}\theta - \frac{\iota}{2\sqrt{2}}\frac{\sin(2\theta)\sin\varphi}{\sqrt{1-\cos\varphi}}\sigma_1(\mathrm{d}\alpha+\mathrm{d}\beta) + \iota\sin\theta\frac{\sigma_3+\sigma_0}{2}(\mathrm{d}\alpha+\mathrm{d}\beta) \\ & + \frac{\iota}{2}\frac{\cos^2\theta\sin^2\varphi}{1-\cos\varphi}\frac{\sigma_0-\sigma_3}{2}(\mathrm{d}\alpha+\mathrm{d}\beta) + \frac{\iota}{2}(1-\cos\varphi)\frac{\sigma_0-\sigma_3}{2}\mathrm{d}\alpha. \end{aligned} \tag{88}$$

The transversal bundle for $\vec{R}=(\alpha,\beta,\theta,\varphi) \in S^4_+$ fixed is the trivial bundle of the dynamics $(\mathbb{R}\times \mathrm{U}(2), \mathbb{R}, \mathrm{U}(2), \mathrm{Pr}_1)$ endowed with the connection

$$\forall t \in \mathbb{R}, E(\vec{R},t) = \frac{\iota}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1-\dfrac{1}{\cos\varphi} \end{pmatrix}\mathrm{d}t. \tag{89}$$

Let χ^S_t be the fiber diffeomorphism of the base bundle $(S, \mathbb{R}, S^4_+, \pi_S)$. By definition we have $P_t=\chi^{S^*}_t P$, but the Hamiltonian H does not have an explicit dependence on t. Then it is clear that $\forall t \in \mathbb{R}$, $P_t=P$ and then χ^S_t is the identity map. We conclude that $\pi^{-1}_S(t)=S^4_+$ and then $S=S^4_+\times\mathbb{R}$. The base bundle is the trivial bundle $(S^4_+\times\mathbb{R}, \mathbb{R}, S^4_+, \mathrm{Pr}_2)$. The local trivializations of the total bundle are

$$\phi^{ij}_{++}(\vec{R},t,g) = \phi^i_P[t](\vec{R},g) = \phi^i_P(\vec{R},g) \tag{90}$$

because P_t is independent of t. Let $\{U^i\times\mathbb{R}\}_{i=NE,NW,SE,SW}$ be the atlas of $S^4_+\times\mathbb{R}$, $\forall(\vec{R},t) \in (U^i\cap U^j)\times\mathbb{R}$ we have the transition functions of the total space P^+ of the total bundle by $g^{ij}_{++}(\vec{R},t)=g^{ij}(\vec{R})$. Then it is clear that $P^+=P\times\mathbb{R}$, the total bundle of the geodynamics is then $(P\times\mathbb{R}, S^4_+\times\mathbb{R}, \mathrm{U}(2), (\pi_P\circ\mathrm{Pr}_1)\times\mathrm{Pr}_2)$. Note that the triviality of the fibration on the time is due to the nonexplicit dependence of H on t. When this is not the case, then the base bundle is not trivial.

D. Different aspects of the quantum dynamical system in our formalism

All the ingredients of the composite bundle formalism have now been explicitly identified for our example. We now want to consider a particular dynamics in order to complete the description of the quantum dynamical system in our composite bundle representation. In order to simplify and to clarify the discussion, we consider a dynamics such that $\forall t$ $\alpha=\beta=0$ (chart U^{NE}) and we use the orginal variables (W,V) in place of (θ,φ); this restricted manifold is denoted by $\mathcal{M}$ in this paragraph. In the sequel $\mu=1,2$, $R^1=W$, $R^2=V$ and $R^0=t$. In these conditions, the gauge potential of the total bundle P^+ over $\mathcal{M}$ is

$$\begin{aligned} A_+ = {} & \iota\hbar^{-1}E(W,V)\mathrm{d}t + A(W,V) = \frac{\iota}{2}(1-\sqrt{1+V^2+W^2})(\sigma_0-\sigma_3)\mathrm{d}t \\ & + \frac{\iota\sigma_2}{\sqrt{2}\sqrt{1+\dfrac{V^2}{W^2}}\sqrt{\dfrac{1+V^2+W^2-\sqrt{1+V^2+W^2}}{V^2}}}\left(\frac{\mathrm{d}W}{W}-\frac{\mathrm{d}V}{V}\right). \end{aligned} \tag{91}$$

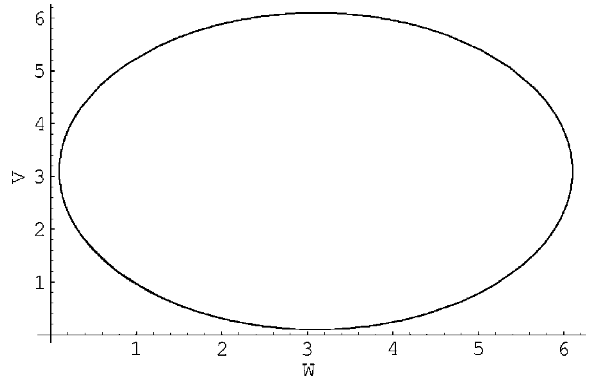

FIG. 3. The path induced by γ in $\mathcal{M}$.

We consider the dynamics described by $\gamma \in \Gamma([t_0=-25,T=90],\mathcal{M}\times\mathbb{R})$ defined by $\gamma(t) = (3\cos(2\pi(t-25)/90)+3.1, 3\sin(2\pi(t-25)/90)+3.1)$ (the units are arbitrary). γ induces a path $\mathcal{C}$ in $\mathcal{M}\times\mathbb{R}$. (See Fig. 3.)

The horizontal lift of $\mathcal{C}$ defines the holonomy operator

$$\forall t \in [t_0,T], \quad J^{\gamma,t_0,t} = \mathbb{P}e^{-\imath\hbar^{-1}\int_{t_0}^{t}E(\gamma(t'))\mathrm{d}t' - \int_0^t A_\mu(\gamma(t'))\mathrm{d}\gamma^\mu(t')/\mathrm{d}t'\mathrm{d}t'} \in \Gamma(\mathcal{M}\times\mathbb{R},P^+). \tag{92}$$

Let $(E^+,\mathcal{M}\times\mathbb{R},\mathbb{C}^2,\pi_{E^+})$ be the associated vector bundle of P^+ by the action of U(2) on $\mathbb{C}^2$ defined by the matrix product. The states of the system are described by the $\mathcal{C}^\infty(\mathcal{M}\times\mathbb{R},\mathbb{C})$-module $\Gamma(\mathcal{M}\times\mathbb{R},E^+)$, which is the space of the sections of E^+. At $t=0$ we suppose that $\psi(0)=(1/\sqrt{2})\times(|1,\gamma(0)\rangle+|2,\gamma(0)\rangle)$; then for all $t\geqslant t_0$

$$\psi(t) = \sum_{b=1,2}\frac{1}{\sqrt{2}}([J^{\gamma,t_0,t}]_{b,1}+[J^{\gamma,t_0,t}]_{b,2})|b,\gamma(t)\rangle \in \Gamma(C,E^+). \tag{93}$$

The state space $\Gamma(\mathcal{M}\times\mathbb{R},E^+)$ is endowed with the $\mathcal{C}^\infty(\mathcal{M}\times\mathbb{R},\mathbb{C})$-valued inner product

$$\forall \chi,\phi \in \Gamma(\mathcal{M}\times\mathbb{R},E^+), \quad \langle\chi|\phi\rangle_{E^+}(\vec{R},t) = \langle\chi(\vec{R},t)|\phi(\vec{R},t)\rangle_{\mathbb{C}^2} \tag{94}$$

$\forall i=1,2$ $|i,\vec{R}\rangle \in \Gamma(\mathcal{M}\times\mathbb{R},E^+)$ [in the composite bundle representation it is the canonical basis $|1\rangle=\binom{1}{0}$ and $|2\rangle=\binom{0}{1}$]. With the scalar product we obtain the instantaneous occupation probabilities of the eigenlevel E_1 and $E_2(V,W)$,

$$P_i(t) = |\langle i|\psi\rangle_{E^+}(\gamma(t),t)|^2. \tag{95}$$

These probabilities are drawn in Fig. 4.

In Sec. IV, we have introduced some fields F_+, B and G in $\mathcal{M}\times\mathbb{R}$ associated with the structure of the composite bundle. An illustration of these fields are shown in Fig. 5.

Let $(V^+,\mathcal{M}\times\mathbb{R},u(2),\pi_{V^+})$ be the associated vector bundle of P^+ by the adjoint action Ad of U(2) on $\mathfrak{u}(2)$ $(\mathrm{Ad}(U)X=U^{-1}XU,\ \forall U\in U(2),\ \forall X\in\mathfrak{u}(2))$. The algebra $\Gamma(\mathcal{M}\times\mathbb{R},V^+)$ endowed with the Lie bracket

$$\forall A,B \in \Gamma(\mathcal{M}\times\mathbb{R},V^+), \quad [A,B]_{V^+}(\vec{R},t) = [A(\vec{R},t),B(\vec{R},t)]_{u(2)} \tag{96}$$

is the observables space. In our example of a three level system, a set of observables has a particular importance. Let $S_i=\frac{1}{2}\lambda_i$ for $i=1,\ldots,8$, and let $S_i(t)=U(t,t_0)S_iU(t,t_0)^\dagger$, where $U(t,t_0)$ is the evolution operator associated with the Schrödinger equation. The role of the set of operators $S_i(t)$ for a three level system has been extensively studied by Ho, Chu *et al.*[21–23] Let ρ_0 be the density matrix of the initial condition of the system. We introduce the vector $\vec{\mathcal{S}}(t)\in\mathbb{R}^8$ such that

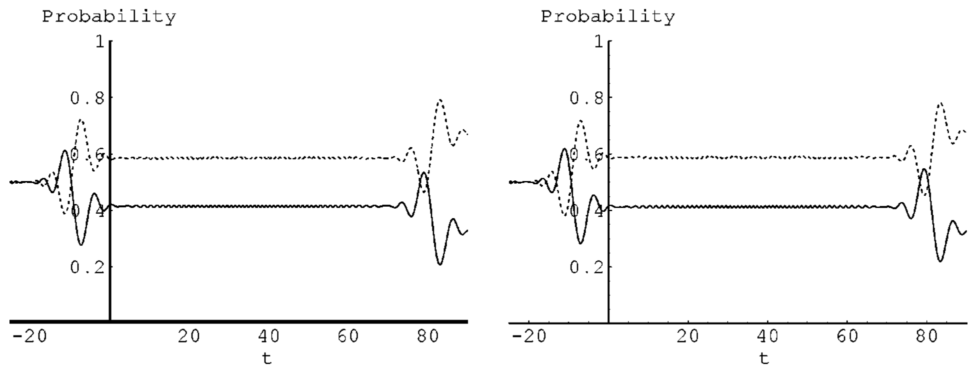

FIG. 4. Left, occupation probabilities of the state $|1\rangle$ (plain line), $|2\rangle$ (dash line), and $|3\rangle$ (strong line) computed by direct integration of the Schrödinger equation in $\mathbb{C}^3$. Right; occupation probabilities of the state $|1\rangle$ (full line), and $|2\rangle$ (dashed line) computed with the formula (93) based on the holonomy operator of the composite bundle. We see that the results obtained by the use of the holonomy operator are in perfect agreement with the direct integration. Moreover the left figure reveals that the level 3 is never occupied, in agreement with its adiabatic elimination in the bundle representation.

$S_i(t)=\mathrm{tr}(\rho_0 S_i(t))$ [the average value of the observable $S_i(t)$]. $\vec{\mathcal{S}}(t)$ is called a coherent vector. From the trajectory of this vector we can obtain information about the dynamical system (for a complete exposition of this subject see Refs. 21–23). Within an approach using our bundle formalism the analogues of the observables $S_i(t)$ are

$$S_i(\vec{R}) = T(\vec{R})^{\dagger} S_i T(\vec{R}) \in \Gamma(\mathcal{M} \times \mathbb{R}, u(2)), \tag{97}$$

and the coherent vector $\vec{\mathcal{S}}(t)$ is obtained by (in our quantum system $\rho_0=|\psi(0)\rangle\langle\psi(0)|$

$$S_i(t) = \langle \psi | S_i \psi \rangle_{E^+}(\gamma(t),t). \tag{98}$$

Figure 6 illustrates the computation of $\mathcal{S}$ in the composite bundle formalism.

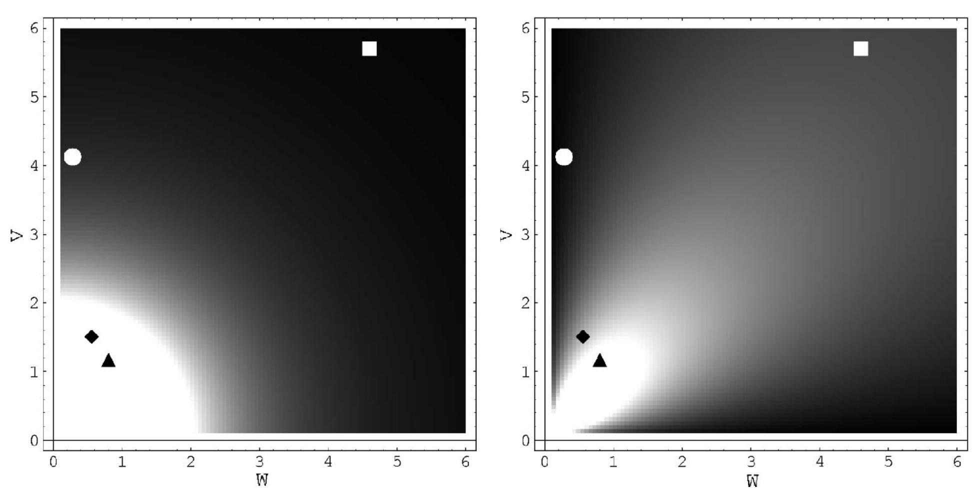

FIG. 5. Left, the (1, 2)-matrix element of $(F_+)_{12}$ with respect to $\mathcal{M}$. Right, the (1, 1)-matrix element of G_{012} with respect to $\mathcal{M}$. The white area is characterized by a strong field intensity whereas the black area corresponds to vanishing fields (arbitrary units). We have moreover indicated some points of the path $\mathcal{C}$, ○, $t=-25$; ◇, $t=-12$; □, $t=40$; and △, $t=80$. By comparison with Fig. 4 we see that the wave function changes significantly only when the control parameters are localized in the strong field area. This shows that these fields are related to the dynamical properties of the quantum system.

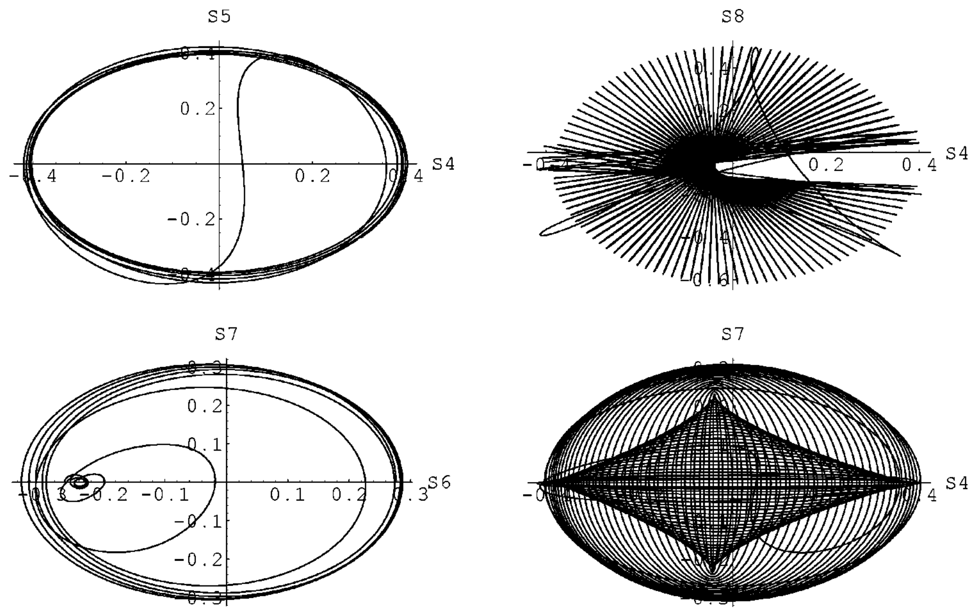

FIG. 6. Trajectories of the coherent vector $\vec{\mathcal{S}}(t)$ projected in different planes, for different time intervals, computed in the composite bundle representation.

The example of the three-level system shows that we can use the composite principal bundle representation to obtain all the physical ingredients of the quantum dynamics. This formalism, coupled with a numerical procedure to compute the holonomy operator, could be used as a powerful method to study a more complex quantum dynamical system.

VI. CONCLUSION

The principal composite bundle appears as a highly appropriate structure to describe the adiabatic transport with a Berry phase which does not commute with the dynamical phase. Nevertheless the use of the standard gauge theory requires us to restrict the gauge transformations to the sections which satisfy the Schrödinger-Von Neumann equation. This feature reveals that it is impossible to describe quantum dynamics with a purely geometric model without a dynamical postulate. If one does not accept any restriction on the gauge transformations, the price to pay is the implementation of an unusual gauge theory which introduces, in addition to the curvature, a field, the curving B, which is precisely the commutator of A with H. It is remarkable that such a situation is very similar to the gauge fields of non-Abelian gerbes, but with the important difference that in the non-Abelian gerbe theory, B does not have values in the Lie algebra g (see Ref. 11). (See Refs. 24–26.)

One can easily generalize this description to the problem of the non-Abelian Aharonov-Anandan phase which does not commute with the dynamical phase; this is done by replacing the principal bundle $(P,\mathcal{M},U(M),\pi_P)$ by the universal principal bundle $(V_M(\mathbb{C}^n),G_M(\mathbb{C}^n),U(M),\pi_U)$. The analysis of Bohm and Mostatazadeh[7] has effectively showed that $(V_M(\mathbb{C}^n),G_M(\mathbb{C}^n),U(M),\pi_U)$ is the universal bundle of $(P,\mathcal{M},U(M),\pi_P)$, and our work demonstrates that the same relationship exists between the adiabatic composite bundle and the universal composite bundle.

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