

A new kind of geometric phases in open quantum systems and higher gauge theory

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Abstract

A new approach is proposed, extending the concept of geometric phases to adiabatic open quantum systems described by density matrices (mixed states). This new approach is based on an analogy between open quantum systems and dissipative quantum systems which uses a C^* -module structure. The gauge theory associated with these new geometric phases does not employ the usual principal bundle structure but a higher structure, a categorical principal bundle (the so-called principal 2-bundle or non-Abelian bundle gerbe) which is sometimes a non-Abelian twisted bundle. The need to site the gauge theory in this higher structure is a geometrical manifestation of the decoherence induced by the environment on the quantum system.

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1. Introduction

Since the pioneering work of Berry [1] concerning the adiabatic dynamics of a closed two-level system, several types of geometric phases for quantum systems have been studied, including non-Abelian geometric phases [2], non-adiabatic geometric phases [3], geometric phases associated with noncyclic evolution [4], geometric phases associated with the Floquet theory [5] and geometric phases associated with resonances [6–8]. The geometric structures within which the Berry phases arise have also been studied, e.g., the line bundle associated with an Abelian Berry phase [9], the principal bundle associated with a non-Abelian Berry phase [2] and the composite bundle associated with a Berry phase which does not commute with the dynamical phase [10–12].

These geometric phase phenomena are related to pure states ψ (vectors in a Hilbert space \mathcal{H}) governed by the Schrödinger equation $i\hbar\frac{d\psi}{dt} = H(t)\psi(t)$ (where $H \in \mathcal{L}(\mathcal{H})$ is the Hamiltonian, which is self-adjoint for the closed quantum systems but not for the dissipative quantum systems). The open quantum systems [13–15] are described by mixed states (density

matrices $\rho \in \mathcal{L}(\mathcal{H})$, $\rho^\dagger = \rho$, $\rho \geq 0$ and $\text{tr}_{\mathcal{H}} \rho = 1$) which are governed by a Lindblad equation $i\hbar \frac{d\rho}{dt} = \mathcal{L}(\rho)$ (where the Lindbladian \mathcal{L} is often written in the standard form: $\mathcal{L}(\rho) = [h, \rho] - \frac{i}{2} \{ \Gamma_k^\dagger \Gamma^k, \rho \} + i \Gamma^k \rho \Gamma_k^\dagger$, $h, \Gamma^k \in \mathcal{L}(\mathcal{H})$, with $\{ \cdot, \cdot \}$ being the anticommutator). Four approaches have previously been proposed to extend the concept of the geometric phases to open quantum systems. Uhlmann's approach [16–19] is based on the purification of mixed states; since a purified state is Hilbert–Schmidt, we can use the definition of the geometric phases for vectors of a Hilbert space. The second approach [20] is based on the decomposition of mixed states into convex combinations of pure states. As the geometric phase associated with a pure state is the holonomy of the natural connection of a universal bundle over a complex projective space, this generalization to mixed states is obtained by considering the natural connection of the bundle induced by the convex combinations. The third approach due to Sarandy and Lidar [21, 22] is based on the fact that in finite dimension ($\dim \mathcal{H} = n$), the reduced density matrices $\rho - I_n$ are Hilbert–Schmidt. Thus, the Lindblad equation can be considered as a Schrödinger equation in an n^2 -dimensional Hilbert space and we can use the definition of the geometric phases for vectors of a Hilbert space. The last approach, due to Sjöqvist *et al* [23, 24], is based on the introduction of a notion phase for evolving mixed states in interferometry. This method is generalized and extended to a nonunitary evolution via a purification method.

In this paper, we propose another approach to define the geometric phases of an open quantum system based on an analogy between open and dissipative quantum systems. This approach uses the identification of a mixed state with a ‘norm’ in a C^* -module (see for example [25]). This identification provides a new interpretation of the geometric phases of open quantum systems. The geometric structure in which this new kind of geometric phase is sited is not a principal bundle (as used for the geometric phases of a closed quantum system) but a higher gauge structure, a non-Abelian bundle gerbe [26–28] or equivalently a principal 2-bundle [29–31], i.e. a categorical generalization of a principal bundle. The higher degree in the gauge theory seems to be a manifestation of the decoherence induced by the environment on which the quantum system is open. In previous works, we have already shown that such a higher degree arises for geometric phases of some quantum systems which are described by pure states but which present a kind of decoherence process due to resonance crossings [8] or to the use of the adiabatic Floquet theory [32].

Section 2 is a brief review concerning geometric phases of closed and dissipative quantum systems. In section 3, we develop an analogy between dissipative quantum systems and open quantum systems, which allows us to introduce in section 4 a new kind of geometric phase for open quantum systems and its related geometric structure. Section 5 presents the adiabatic approximation for open quantum systems from the viewpoint of the geometric phase approach introduced in section 4. Section 6 presents a very simple example: the control of a qubit subjected to a decoherence process. For the sake of simplicity, this paper focuses on finite-dimensional open quantum systems.

2. Geometric phases of closed and dissipative quantum systems

2.1. Geometric phases of closed quantum systems

Let $(\mathcal{H}, H(x), M)$ be a quantum dynamical system, where \mathcal{H} is a Hilbert space representing the quantum state space (for the sake of simplicity, we suppose a finite-dimensional Hilbert space, $\dim \mathcal{H} = n$), $M \ni x \mapsto H(x) \in \mathcal{L}(\mathcal{H})$ is a continuous family of self-adjoint operators representing the driven Hamiltonian and M is a C^∞ -manifold representing the space of all possible configurations of the external classical parameters controlling the quantum system. A

given dynamics is generated by a control parameter evolution $t \mapsto x(t)$ as being the solution of the Schrödinger equation

$$i\hbar \frac{d\psi(t)}{dt} = H(x(t))\psi(t), \quad \psi(0) = \psi_0 \in \mathcal{H}. \quad (1)$$

Let $\{U^\alpha\}_\alpha$ be a good open cover of M (a set of contractible open sets such that $\bigcup_\alpha U^\alpha = M$). Let $M \ni x \mapsto e(x) \in \mathbb{R}$ be a continuous eigenvalue of $H(x)$ supposed non-degenerate and $U^\alpha \ni x \mapsto \phi_e^\alpha(x) \in \mathcal{H}$ be the associated (locally defined) continuous normalized eigenvector (the eigenvector is only locally defined because in many cases there do not exist eigenvectors continuously extendable on the whole of M):

$$\forall x \in U^\alpha, \quad H(x)\phi_e^\alpha(x) = e(x)\phi_e^\alpha(x). \quad (2)$$

If we suppose that $\psi(0) = \phi_e^\alpha(x(0))$ and that the evolution $t \mapsto x(t)$ is slow with respect to the quantum proper time of state transition, i.e.

$$\begin{aligned} \forall f \in \text{Sp}(H) \setminus \{e\}, \quad \forall \phi_f \in \ker(H - f1_{\mathcal{H}}) \\ \langle \phi_f(x(t)) | \frac{d}{dt} \phi_e(x(t)) \rangle = \frac{\langle \phi_f(x(t)) | \frac{dH(x(t))}{dt} | \phi_e(x(t)) \rangle}{e(x(t)) - f(x(t))} \simeq 0 \end{aligned} \quad (3)$$

(note that this requires a gap condition $|e(x(t)) - f(x(t))| > 0, \forall t, \forall f \in \text{Sp}(H) \setminus \{e\}$), then the wavefunction is

$$\psi(t) \simeq e^{-i\hbar^{-1} \int_0^t e(x(t')) dt'} e^{-\int_{x(0)}^{x(t)} c^{A^\alpha}} \phi_e^\alpha(x(t)), \quad (4)$$

where \mathcal{C} is the path in M drawing by $t \mapsto x(t)$ and

$$A^\alpha = \langle \phi_e^\alpha | d\phi_e^\alpha \rangle \in \Omega^1(U^\alpha, i\mathbb{R}) \quad (5)$$

is the generator of the geometric phase (d is the exterior derivative of M and $\Omega^n(M, i\mathbb{R})$ is the set of pure imaginary-valued differential n -forms of M). We have supposed that \mathcal{C} is completely included in U^α . Equation (4) is called the adiabatic transport formula, $e^{-i\hbar^{-1} \int_0^t e(x(t')) dt'}$ is called the dynamical phase and $e^{-\int_{x(0)}^{x(t)} c^{A^\alpha}}$ is the geometric phase which depends only on the shape of \mathcal{C} in M and not on the time duration. By construction, the wavefunction $\tilde{\psi}(t) = e^{-\int_{x(0)}^{x(t)} c^{A^\alpha}} \phi_e^\alpha(x(t))$ satisfies the parallel transport condition $\langle \tilde{\psi}(t) | \frac{d\tilde{\psi}(t)}{dt} \rangle = 0$.

If \mathcal{C} crosses several charts of $\{U^\alpha\}_\alpha$, we have

$$\psi(t) \simeq e^{-i\hbar^{-1} \int_0^t e(x(t')) dt'} e^{-\int_{x(0)}^{x^\alpha} c^{A^\alpha}} g^{\beta\alpha}(x^{\alpha\beta}) e^{-\int_{x^\alpha}^{x^\beta} c^{A^\beta}} g^{\xi\alpha}(x^{\alpha\xi}) e^{-\int_{x^\beta}^{x^\xi} c^{A^\xi}} \phi_e^\xi(x(t)), \quad (6)$$

where $U^\alpha, U^\beta, \dots, U^\xi, U^\zeta$ are the different charts crossed by \mathcal{C} , $x^{\alpha\beta}$ is an arbitrary point in $U^\alpha \cap U^\beta \cap \mathcal{C}$ and $g^{\alpha\beta}(x) \in U(1)$ is defined by

$$\forall x \in U^\alpha \cap U^\beta \quad \phi_e^\beta(x) = g^{\alpha\beta}(x)\phi_e^\alpha(x) \quad (7)$$

(since over $U^\alpha \cap U^\beta$, ϕ_e^α and ϕ_e^β are two normalized eigenvectors of the same non-degenerate eigenvalue, they differ only by a phase factor $g^{\alpha\beta}$). The formula of the geometric phase is independent of the arbitrary choices of transition points $\{x^{\alpha\beta}\}_{\alpha,\beta}$ [33]. The transition functions $g^{\alpha\beta}$ satisfy the cocycle relation

$$\forall x \in U^\alpha \cap U^\beta \cap U^\gamma, \quad g^{\alpha\beta}(x)g^{\beta\gamma}(x)g^{\gamma\alpha}(x) = 1, \quad (8)$$

which is required in order to have a single-valued eigenvector $\phi_e^\alpha = g^{\alpha\beta}g^{\beta\gamma}g^{\gamma\alpha}\phi_e^\alpha$.

As pointed out by Simon [9], the geometric phase is associated with the horizontal lift of \mathcal{C} in P , the principal $U(1)$ -bundle over M defined by the transition functions $g^{\alpha\beta}$ and endowed with the connection having A^α as gauge potential (see [34] for an exposition of the principal bundle theory). If the path \mathcal{C} is closed, we have at the end T of the evolution

$$\psi(T) \simeq e^{-i\hbar^{-1} \int_0^T e(x(t')) dt'} \text{hol}(\mathcal{C})\phi_e^\alpha(x(0)), \quad (9)$$

where $\text{hol}(\mathcal{C})$ is the holonomy of the horizontal lift of \mathcal{C} in P which (if $\mathcal{C} \subset U^\alpha$) is

$$\text{hol}(\mathcal{C}) = e^{\int_{\mathcal{C}} A^\alpha}. \quad (10)$$

The geometry of the principal bundle P is characterized by the curvature $F = dA^\alpha \in \Omega^2(M, i\mathbb{R})$ which satisfies $dF = 0$ (since $dA^\alpha = dA^\beta$, for all $x \in U^\alpha \cap U^\beta$ the curvature F is globally defined).

We now suppose that e_a is an m -fold degenerate eigenenergy on the whole of M . Let $\{\phi_{e,a}^\alpha\}_{a=1,\dots,m}$ be an orthonormal basis of continuous associated eigenvectors. If $\psi(0) = \phi_{e,a}^\alpha(x(0))$ and $\mathcal{C} \subset U^\alpha$, then at the adiabatic limit we have

$$\psi(t) \simeq e^{-i\hbar^{-1} \int_0^t e(x(t')) dt'} \sum_{b=1}^m [\mathbb{P}_{\mathcal{C}} e^{-\int_{x(0)}^{x(t)} A^\alpha}]_{ba} \phi_{e,b}^\alpha(x(t)), \quad (11)$$

where

$$A^\alpha = \begin{pmatrix} \langle \phi_{e,1}^\alpha | d\phi_{e,1}^\alpha \rangle & \dots & \langle \phi_{e,1}^\alpha | d\phi_{e,m}^\alpha \rangle \\ \vdots & \ddots & \vdots \\ \langle \phi_{e,m}^\alpha | d\phi_{e,1}^\alpha \rangle & \dots & \langle \phi_{e,m}^\alpha | d\phi_{e,m}^\alpha \rangle \end{pmatrix} \in \Omega^1(U^\alpha, \mathfrak{u}(m)), \quad (12)$$

with $\Omega^n(U^\alpha, \mathfrak{u}(m))$ the set of $\mathfrak{u}(m)$ valued n -differential forms of M , where $\mathfrak{u}(m)$ is the Lie algebra of the anti-self-adjoint matrices of order m , and $\mathbb{P}_{\mathcal{C}} e^{-\int_{x(0)}^{x(t)} A^\alpha}$ is the path-ordered exponential along \mathcal{C} [34], i.e. $\mathbb{P}_{\mathcal{C}} e^{-\int_{x(0)}^{x(t)} A}$ is solution of

$$\frac{d\mathbb{P}_{\mathcal{C}} e^{-\int_{x(0)}^{x(t)} A}}{dt} = -A_\mu(x(t)) \dot{x}^\mu(t) \mathbb{P}_{\mathcal{C}} e^{-\int_{x(0)}^{x(t)} A}. \quad (13)$$

The non-Abelian geometric phase $\mathbb{P}_{\mathcal{C}} e^{-\int_{x(0)}^{x(t)} A^\alpha}$ is then associated with the horizontal lift of \mathcal{C} in P , the principal $U(m)$ -bundle on the right over M ($U(m)$ is the Lie group of the unitary matrices of order m) defined by the transition functions $g^{\alpha\beta}(x) \in U(m)$ such that

$$\forall x \in U^\alpha \cap U^\beta, \quad [g^{\alpha\beta}(x)]_{ab} = \langle \phi_{e,a}^\alpha(x) | \phi_{e,b}^\beta(x) \rangle, \quad (14)$$

where $g^{\alpha\beta}$ is the passage matrix between the two bases $\{\phi_{e,a}^\alpha\}_a$ and $\{\phi_{e,a}^\beta\}_a$. P is endowed with the connection having A^α as gauge potential, and for a closed path $\mathbb{P}_{\mathcal{C}} e^{\int_{\mathcal{C}} A^\alpha}$ is the holonomy of the horizontal lift of \mathcal{C} in P . The geometry of P is characterized by the curvature $F^\alpha = dA^\alpha + A^\alpha \wedge A^\alpha \in \Omega^2(U^\alpha, \mathfrak{u}(m))$ which satisfies $F^\beta(x) = g^{\alpha\beta}(x)^{-1} F^\alpha(x) g^{\alpha\beta}(x)$ ($\forall x \in U^\alpha \cap U^\beta$) and the Bianchi identity $dF^\alpha + [A^\alpha, F^\alpha] = 0$.

2.2. Geometric phases of dissipative quantum systems

Let $(\mathcal{H}, H(x), M)$ be a quantum dynamical system where $H(x)$ is not self-adjoint with $\frac{1}{2}(H(x) - H(x)^\dagger) \leq 0$. The system is then a dissipative quantum system; the eigenvalues of $H(x)$ are complex with a negative imaginary part (they are called resonances). Let $t \mapsto \psi(t)$ be the wavefunction solution of the Schrödinger equation. The non-self-adjointness of H implies

$$\|\psi(0)\|^2 = 1 \quad \forall t > 0, \quad \|\psi(t)\|^2 < 1. \quad (15)$$

The decreasing of $\|\psi(t)\|^2$ is the characteristic of the dissipation of the quantum system.

Let $M \ni x \mapsto e(x) \in \mathbb{C}$ be a continuous supposed non-degenerate eigenvalue of $H(x)$ and $U^\alpha \ni x \mapsto \phi_e^\alpha(x) \in \mathcal{H}$ be the associated (locally defined) continuous (not necessary normalized) eigenvector. In a similar way to that for the closed case, we can define at the

adiabatic limit a ‘dissipative geometric phase’¹, $e^{-\int_{x(0)}^{x(t)} c A^\alpha} \in \mathbb{C}^*$ by defining the geometric phase generator as

$$A^\alpha = \frac{\langle \phi_e^\alpha | d\phi_e^\alpha \rangle}{\|\phi_e^\alpha\|^2} \in \Omega^1(U^\alpha, \mathbb{C}). \tag{16}$$

Dissipative geometric phases are associated with the horizontal lifts in a principal \mathbb{C}^* -bundle P .

The gauge group \mathbb{C}^* must be viewed as the product $\mathbb{C}^* = U(1) \times \mathbb{R}^{*+}$ of two physical gauge groups (and consequently P must be viewed as the product of two bundles). $U(1)$ is associated with the ordinary geometric phase acquired by the wavefunction during the evolution, which is completely equivalent to the geometric phase of a closed quantum system, whereas \mathbb{R}^{*+} is associated with the geometric contribution to the dissipation. If we are interested only in the dissipation process, we can describe just the structure involving the gauge group \mathbb{R}^{*+} .

The main object is then $\|\psi\|^2 \in \mathbb{R}^{*+}$ which plays the role of the state of dissipation of the quantum system. We call it the density of the system. The Schrödinger equation induces the following dynamical equation for the dissipation:

$$i\hbar \frac{d\|\psi\|^2}{dt} = \mathcal{L}_{x(t)}(\|\psi\|^2), \tag{17}$$

with

$$\mathcal{L}_x(\|\psi\|^2) = \langle \psi | (H(x) - H(x)^\dagger) | \psi \rangle. \tag{18}$$

The eigenvalue equation induces the following equation for the eigendensity:

$$\mathcal{L}_x(\|\phi_e^\alpha(x)\|^2) = 2t \operatorname{Im}(e(x)) \|\phi_e^\alpha(x)\|^2. \tag{19}$$

The generator of the density geometric phase (the geometric contribution to the dissipation) is defined as

$$\operatorname{Re} A^\alpha = \frac{1}{2} \frac{d\|\phi_e^\alpha\|^2}{\|\phi_e^\alpha\|^2} \in \Omega^1(U^\alpha, \mathbb{R}). \tag{20}$$

$\operatorname{Re} A^\alpha$ is then the gauge potential of the connection of a principal \mathbb{R}^{*+} -bundle.

The geometry of the density geometric phases is trivial since the connection is pure gauge, i.e. the gauge potential is d -exact:

$$\operatorname{Re} A^\alpha = d \ln(\|\phi_e^\alpha\|). \tag{21}$$

Consequently, each holonomy is reduced to 1:

$$\operatorname{hol}(\mathcal{C}) = e^{\int_{\mathcal{C}} \operatorname{Re} A^\alpha} = e^{\int_{\mathcal{C}} d \ln(\|\phi_e^\alpha\|)} = 1. \tag{22}$$

Although the geometry of the density geometric phases of dissipative quantum systems is trivial, we use it as a model to construct a theory for open quantum systems which is not trivial.

¹ Since the dissipative geometric phase is not unitary, another terminology, as a geometric factor for example, could be more appropriate.

3. C^* -module of an open quantum system, eigenvectors and gauge invariances

3.1. The C^* -module modelling an open quantum system

An open quantum system consists of a quantum system \mathcal{S} which can exchange matter and/or energy with an environment \mathcal{E} . The total closed system $\mathcal{S} + \mathcal{E}$ is called the universe. Being a closed system, the universe can be described as a quantum dynamical system $(\mathcal{H}_S \otimes \mathcal{H}_E, H(x), M)$ where \mathcal{H}_S is the Hilbert space of the quantum states of the system (which is supposed finite dimensional, $\dim \mathcal{H}_S = n$) and \mathcal{H}_E is the Hilbert space of the quantum states of the environment. To avoid any confusion, the scalar product in \mathcal{H}_S is denoted by $\langle \cdot | \cdot \rangle_{\mathcal{H}_S}$, the scalar product in \mathcal{H}_E is denoted by $\langle \cdot | \cdot \rangle_{\mathcal{H}_E}$ and the scalar product in $\mathcal{H}_S \otimes \mathcal{H}_E$ is denoted by $\langle\langle \cdot | \cdot \rangle\rangle$. In the same manner, the adjoint with respect to $\langle \cdot | \cdot \rangle_{\mathcal{H}_S}$ is denoted by \dagger , whereas the adjoint with respect to $\langle\langle \cdot | \cdot \rangle\rangle$ is denoted by \ddagger . The Hamiltonian of the universe $H(x) \in \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$ (which is self-adjoint for $\langle\langle \cdot | \cdot \rangle\rangle$) can be written as

$$H(x) = H_S(x) \otimes 1_{\mathcal{H}_E} + 1_{\mathcal{H}_S} \otimes H_E(x) + H_I(x), \quad (23)$$

where $H_S \in \mathcal{L}(\mathcal{H}_S)$ is the Hamiltonian of the isolated system, $H_E \in \mathcal{L}(\mathcal{H}_E)$ is the Hamiltonian of the environment and $H_I \in \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$ is the system–environment interaction operator (all these operators are self-adjoint for the scalar products of their respective Hilbert spaces). The mixed state ρ of the system associated with the normalized pure state ψ of the universe is defined as being the partial trace of the orthogonal projector on ψ :

$$\rho = \text{tr}_{\mathcal{H}_E}(|\psi\rangle\langle\psi|), \quad (24)$$

where ρ is called a density matrix and we denote by $\mathcal{D}(\mathcal{H}_S) = \{\rho \in \mathcal{L}(\mathcal{H}_S) | \rho^\dagger = \rho, \rho \geq 0, \text{tr}_{\mathcal{H}_S} \rho = 1\}$ the set of density matrices. ρ can be viewed as a statistical convex combination of pure states, $\rho = \sum_{i=1}^n p_i |\chi_i\rangle\langle\chi_i|$, where $\{p_i\}_i = \text{Sp}(\rho)$ ($0 \leq p_i \leq 1$) and $\{\chi_i \in \mathcal{H}_S\}_i$ are the associated eigenvectors. The Schrödinger equation for the environment induces the Schrödinger–von Neuman equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle\langle\psi(t)| = [H(x(t)), |\psi(t)\rangle\langle\psi(t)|], \quad (25)$$

which induces by the partial trace the Lindblad equation

$$i\hbar \frac{d\rho(t)}{dt} = \mathcal{L}_{x(t)}(\rho(t)), \quad (26)$$

where

$$\begin{aligned} \mathcal{L}_x(\rho) &= [H_S(x), \rho] + \text{tr}_{\mathcal{H}_E}([H_E(x) + H_I(x), |\psi\rangle\langle\psi|]) \\ &= [H_S(x), \rho] + \mathcal{D}_x(\rho). \end{aligned} \quad (27)$$

The Liouvillian $[H_S(x), \cdot]$ represents the free evolution of the system, whereas \mathcal{D}_x represents the decoherence induced by the environment on the system.

A dissipative quantum system can be viewed as a kind of open quantum system in which the dissipation is the only one effect of the coupling with the environment. The density $\|\psi\|^2 \in \mathbb{R}^{+*}$ of equation (15) and the density matrix $\rho \in \mathcal{D}(\mathcal{H}_S)$ of equation (24) play then the same role. They characterize the influence of the environment on the system. The equivalent of \mathcal{H}_S for a dissipative quantum system is \mathbb{C} , the equivalent of \mathcal{H}_E is then \mathcal{H} and the equivalent of the state space of the universe $\mathcal{H}_S \otimes \mathcal{H}_E$ is then $\mathbb{C} \otimes \mathcal{H} = \mathcal{H}$.

We can then consider the open quantum systems in the same manner as the dissipative quantum systems, and use the following paradigm: we consider $\mathcal{H}_S \otimes \mathcal{H}_E$ no longer as a vector space over the ring \mathbb{C} but as a left C^* -module over the C^* -algebra $\mathcal{L}(\mathcal{H}_S)$ (a module which

has the same axioms as a vector space but where an algebra takes the place of \mathbb{C} , see [25]). The C^* -module $\mathcal{H}_S \otimes \mathcal{H}_E$ is endowed with the following inner product:

$$\begin{aligned} (\mathcal{H}_S \otimes \mathcal{H}_E) \times (\mathcal{H}_S \otimes \mathcal{H}_E) &\rightarrow \mathcal{L}(\mathcal{H}_S) \\ (\psi, \phi) &\mapsto \langle \psi | \phi \rangle_* = \text{tr}_{\mathcal{H}_E} (|\phi\rangle\langle\psi|). \end{aligned} \quad (28)$$

This inner product has the following useful properties:

- it is linear on the right and antilinear on the left: $\forall A, B \in \mathcal{L}(\mathcal{H}_S), \forall \phi, \psi, \chi \in \mathcal{H}_S \otimes \mathcal{H}_E$,

$$\langle \psi | A\phi + B\chi \rangle_* = A\langle \psi | \phi \rangle_* + B\langle \psi | \chi \rangle_* \quad (29)$$

$$\langle A\psi + B\chi | \phi \rangle_* = \langle \psi | \phi \rangle_* A^\dagger + \langle \chi | \phi \rangle_* B^\dagger; \quad (30)$$

- it is ‘Hermitian’: $\forall \phi, \psi \in \mathcal{H}_S \otimes \mathcal{H}_E$,

$$\langle \phi | \psi \rangle_* = \langle \psi | \phi \rangle_*^\dagger; \quad (31)$$

- it is ‘positive definite’: $\forall \psi \in \mathcal{H}_S \otimes \mathcal{H}_E$,

$$\langle \psi | \psi \rangle_* \in \mathcal{D}(\mathcal{H}_S) \times \mathbb{R}^{*+} \quad (32)$$

$$\langle \psi | \psi \rangle_* = 0 \iff \psi = 0. \quad (33)$$

A mixed state of the system (a density matrix) is then the square $*$ -norm of a normalized state of the universe:

$$\rho = \|\psi\|_*^2. \quad (34)$$

The analogy between open and dissipative quantum systems is then complete.

Remark. The Hamiltonian of a dissipative system is not self-adjoint; for an open system, even if H is self-adjoint for the scalar product $\langle\langle \cdot | \cdot \rangle\rangle$, it is not self-adjoint for the inner product $\langle \cdot | \cdot \rangle_*$.

3.2. Eigenoperator equation in the C^* -module

The replacement of \mathbb{C} by $\mathcal{L}(\mathcal{H}_S)$ requires the introduction of a new definition of the eigenvectors.

Definition 1 (Eigenoperator and $*$ -eigenvector). $M \ni x \mapsto E(x) \in \mathcal{L}(\mathcal{H}_S)$ is said to be a continuous eigenoperator of $H(x)$ and $U^\alpha \ni x \mapsto \phi_E^\alpha(x) \in \mathcal{H}_S \otimes \mathcal{H}_E$ is said to be an associated (locally defined) continuous $*$ -eigenvector if

$$H(x)\phi_E^\alpha(x) = E(x)\phi_E^\alpha(x) \quad (35)$$

with

$$[E(x) \otimes 1_{\mathcal{H}_E}, H(x)] = 0, \quad (36)$$

$x \mapsto E(x)$ being a continuous map on M and $x \mapsto \phi_E^\alpha(x)$ being a continuous map on U^α .

This definition is the exact transposition of an eigenvalue equation except that E is an operator belonging to the C^* -algebra $\mathcal{L}(\mathcal{H}_S)$. We note that the commutation between E and H is required to have a behaviour sufficiently close to an eigenvalue. We suppose moreover that $\langle\langle \phi_E^\alpha | \phi_E^\alpha \rangle\rangle = 1$ in order to ensure that $\rho_E^\alpha = \|\phi_E^\alpha\|_*^2 = \text{tr}_{\mathcal{H}_E} (|\phi_E^\alpha\rangle\langle\phi_E^\alpha|)$ is a mixed state (the mixed eigenstate). The eigenoperator satisfies some interesting mathematical properties which are set out in appendix A; one of them permits an easy solution of the eigenoperator equation.

3.3. Gauge invariances of the eigenoperator equation

For the usual eigenvalue equation $H\phi_e = e\phi_e$ (with $e \in \mathbb{C}$ non-degenerate), the gauge invariance is very simple; the action of the group \mathbb{C}^* (or $U(1)$ if we want to preserve the normalization) transforms an eigenvector associated with e to another eigenvector associated with e . The situation is more complicated for the eigenoperator equation (35).

Let $G_x \subset \mathcal{GL}(\mathcal{H}_S)$ be the maximal subgroup of $\mathcal{GL}(\mathcal{H}_S)$ ($\mathcal{GL}(\mathcal{H}_S)$ is the group of invertible operators of \mathcal{H}_S) which leaves invariant the vector subspace $\ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E}) \subset \mathcal{H}_S \otimes \mathcal{H}_E$:

$$G_x \ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E}) \subset \ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E}). \quad (37)$$

Let $K_x \subset \mathcal{U}(\mathcal{H}_E)$ be the isotropy subgroup of $H(x)$ within $\mathcal{U}(\mathcal{H}_E)$ ($\mathcal{U}(\mathcal{H}_E)$ is the group of unitary operators of \mathcal{H}_E):

$$K_x = \{k \in \mathcal{U}(\mathcal{H}_E) | k^{-1}H(x)k = H(x)\} \equiv \mathcal{U}(\mathcal{H}_E)_{H(x)}. \quad (38)$$

$G_x \times K_x$ constitutes the gauge group leaving invariant the eigenoperator equation; $\forall gk \in G_x \times K_x$, we have

$$H(x)\phi_E^\alpha(x) = E(x)\phi_E^\alpha(x) \Rightarrow H(x)gk\phi_E^\alpha(x) = E(x)gk\phi_E^\alpha(x). \quad (39)$$

This follows directly from the definitions of G_x and K_x and from the fact that $kE = Ek$ since $k \in \mathcal{L}(\mathcal{H}_E)$ and $E \in \mathcal{L}(\mathcal{H}_S)$.

We denote by \mathfrak{g}_x the Lie algebra of G_x (the maximal subalgebra of $\mathcal{L}(\mathcal{H}_S)$ leaving invariant $\ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E})$) and by $\mathfrak{k}_x = \{k \in \mathcal{L}(\mathcal{H}_E) | k^\dagger = -k, [1_{\mathcal{H}_S} \otimes k, H(x)] = 0\}$ the Lie algebra of K_x .

Property 1. Under a gauge transformation $\tilde{\phi}_E^\alpha(x) = gk\phi_E^\alpha(x)$ with $gk \in G_x \times K_x$, the mixed eigenstate is transformed as follows:

$$\tilde{\rho}_E^\alpha(x) = g\rho_E^\alpha(x)g^\dagger. \quad (40)$$

Proof.

$$\begin{aligned} \tilde{\rho}_E &= \text{tr}_{\mathcal{H}_E}(|\tilde{\phi}_E\rangle\langle\tilde{\phi}_E|) \\ &= \text{tr}_{\mathcal{H}_E}(gk|\phi_E\rangle\langle\phi_E|k^{-1}g^\dagger) \\ &= g \text{tr}_{\mathcal{H}_E}(k^{-1}k|\phi_E\rangle\langle\phi_E|)g^\dagger \\ &= g\rho_E g^\dagger. \end{aligned} \quad (41)$$

□

G_x is the maximal subgroup of $\mathcal{GL}(\mathcal{H}_S)$ leaving equivariant the square $*$ -norm. From the viewpoint of the analogy between open and dissipative quantum systems, a G_x -gauge transformation is equivalent to a norm change of the eigenvector. The property for the dissipative quantum systems, $\forall \lambda \in \mathbb{C}^*$, $\tilde{\phi}_e^\alpha = \lambda\phi_e^\alpha \Rightarrow \|\tilde{\phi}_e^\alpha\|^2 = |\lambda|^2 \|\phi_e^\alpha\|^2$, becomes for the open quantum systems, $\forall g \in G_x$, $\tilde{\phi}_E^\alpha = g\phi_E^\alpha \Rightarrow \|\tilde{\phi}_E^\alpha\|_*^2 = g\|\phi_E^\alpha\|_*^2 g^\dagger$. Since K_x leaves invariant the square $*$ -norm, a K_x -gauge transformation is a kind of ‘phase change’ specific to the open quantum system.

The definition of $G_x \times K_x$, the group equivalent to a norm change for a $*$ -eigenvector, permits us to define the notion of degeneracy for the eigenoperator.

Definition 2 (Non-degenerate eigenoperator). Let $M \ni x \mapsto E(x) \in \mathcal{L}(\mathcal{H}_S)$ be an eigenoperator of $H(x)$ and $U^\alpha \ni x \mapsto \phi_E^\alpha(x) \in \mathcal{H}_S \otimes \mathcal{H}_E$ be an associated $*$ -eigenvector. Let $G_x \times K_x$ be the gauge group defined as previously. We say that $E(x)$ is non-degenerate on U^α if $\forall x \in U^\alpha$ the action of $G_x \times K_x$ on $\ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E})$ is transitive, i.e. if

$$G_x \times K_x \phi_E^\alpha(x) = \ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E}). \quad (42)$$

We see that K_x has no direct influence on ρ_E . We introduce another gauge group J_x related to K_x (this relation will be apparent in the next section) but which is associated with ρ_E . Let $J_x^0 \subset G_x$ be the union of all isotropy subgroups of the elements of the orbit by G_x of $\rho_E^\alpha(x)$:

$$\begin{aligned} J_x^0 &= \{j \in G_x | \exists g \in G_x, jg\rho_E^\alpha(x)g^\dagger j^\dagger = g\rho_E^\alpha(x)g^\dagger\} \\ &= \bigcup_{g \in G_x} G_{x, g\rho_E^\alpha(x)g^\dagger} \\ &= \bigcup_{g \in G_x} gG_{x, \rho_E^\alpha(x)}g^{-1} \end{aligned} \quad (43)$$

($G_{x, \rho_E^\alpha(x)} = \{j \in G_x | j\rho_E^\alpha(x)j^\dagger = \rho_E^\alpha(x)\}$). J_x^0 is then also the orbit of the isotropy subgroup of ρ_E^α by the action of G_x on itself. J_x^0 is a normal subgroup of G_x (see appendix A).

Suppose that $\dim \ker(\rho_E^\alpha(x)) = n - p$; let $\mathfrak{j}_x^1 \subset \mathfrak{g}_x$ be

$$\mathfrak{j}_x^1 = \left\{ O \in \mathfrak{g}_x | \hat{O} = \begin{pmatrix} 0_{p \times p} & *_{p \times (n-p)} \\ 0_{(n-p) \times p} & *_{(n-p) \times (n-p)} \end{pmatrix} \right\}, \quad (44)$$

where \hat{O} is the matrix representation of O in the diagonalization basis of ρ_E^α (the eigenvectors associated with the eigenvalue 0 being from the position $n - p$ to the position n), $0_{n \times p}$ is the null $n \times p$ matrix and $*_{n \times p}$ denotes any $n \times p$ matrix. \mathfrak{j}_x^1 is a normal solvable subalgebra of \mathfrak{g}_x (see appendix A).

Let $J_x^1 = \{e^j, j \in \mathfrak{j}_x^1\}$ be the Lie group of \mathfrak{j}_x^1 . J_x is the direct product of groups $J_x = J_x^0 \times J_x^1$. We denote by $\mathfrak{j}_x = \mathfrak{j}_x^0 \oplus \mathfrak{j}_x^1$ its Lie algebra, $\mathfrak{j}_x^0 = \{j \in \mathfrak{g}_x | \exists g \in G, jg\rho_E^\alpha(x)g^\dagger + g\rho_E^\alpha(x)g^\dagger j^\dagger = 0\}$. Noting that if $\rho_E^\alpha(x)$ is invertible, then J_x is reduced to J_x^0 .

In order to have the same behaviour on the whole of M , we assume for the rest of this paper the following *local triviality assumption*:

- $\forall x, y \in M$, G_x and G_y are isomorphic,
- $\forall x, y \in M$, $\dim \ker(\rho_E^\alpha(x)) = \dim \ker(\rho_E^\alpha(y))$ (and by consequences J_x and J_y are isomorphic).

Let G be the abstract group typical of the family $\{G_x\}_{x \in M}$ and $\forall x \in U^\alpha$, let $\zeta_x^\alpha : G \rightarrow G_x \subset \mathcal{GL}(\mathcal{H}_S)$ be the group isomorphism between G and G_x continuous with respect to x (ζ_x^α is locally defined because the existence of a continuous extension on the whole of M is not ensured). $\forall x, y \in U^\alpha$, $\zeta_y^\alpha \circ (\zeta_x^\alpha)^{-1}$ constitutes the isomorphism between G_x and G_y . Let J be the subgroup of G such that $\zeta_x^\alpha(J) = J_x$ (by construction J is the product of a normal and a solvable subgroup of G). The map ζ_x^α defines a principal G -bundle on the left over U^α as being its fibre diffeomorphism. We denote by P_G^α this bundle and by $\zeta^\alpha : U^\alpha \times G \rightarrow P_G^\alpha$ its local trivialization defined by $\zeta^\alpha(x, g) = \zeta_x^\alpha(g)$. The restriction of ζ^α on $U^\alpha \times J$ defines a principal J -bundle on the left over U^α denoted by P_J^α . The bundles P_G^α and P_J^α encode the gauge theory associated with the eigenoperator equation. We denote by $P_{\mathfrak{g}}^\alpha = P_G^\alpha \times_G \mathfrak{g}$ and by $P_{\mathfrak{j}}^\alpha = P_J^\alpha \times_J \mathfrak{j}$ the vector bundles over U^α associated with P_G^α and P_J^α , respectively, by the adjoint representation of G on \mathfrak{g} , $\text{Ad}(g)X = gXg^{-1}$ (\mathfrak{g} and \mathfrak{j} are the Lie algebras of G and J , respectively). In a same manner, we define the bundles P_K^α and $P_{\mathfrak{k}}^\alpha$.

We consider now the gauge transformations at the intersection of several charts. Let $g^{\alpha\beta} \in \mathcal{C}^\infty(U^\alpha \cap U^\beta) \otimes_M \bigsqcup_x G_x$ and $k^{\alpha\beta} \in \mathcal{C}^\infty(U^\alpha \cap U^\beta) \otimes_M \bigsqcup_x K_x$ be the 1-transition functions defined as follows: $\forall x \in U^\alpha \cap U^\beta$,

$$\phi_E^\alpha(x) = k^{\alpha\beta}(x)g^{\alpha\beta}(x)\phi_E^\beta(x) \quad (45)$$

and then

$$\rho_E^\alpha(x) = g^{\alpha\beta}(x)\rho_E^\beta(x)g^{\alpha\beta}(x)^\dagger \quad (46)$$

(\otimes_M denotes the image of the tensor product by the diagonal map which transforms $(x, y) \mapsto f(x, y)$ to $x \mapsto f(x, x)$ and \sqcup denotes the disjoint union).

The 1-transition functions generate some 2-transition functions $h^{\alpha\beta\gamma} \in \mathcal{C}^\infty(U^\alpha \cap U^\beta \cap U^\gamma) \otimes_M \bigsqcup_x J_x$ measuring the failure of the cocycle relation (which is not ensured): $\forall x \in U^\alpha \cap U^\beta \cap U^\gamma$,

$$h^{\alpha\beta\gamma}(x) = g^{\alpha\beta}(x)g^{\beta\gamma}(x)g^{\gamma\alpha}(x), \quad (47)$$

where $h^{\alpha\beta\gamma}(x)$ is an element of J_x since to have a single-valued mixed state, we must have $g^{\alpha\beta}g^{\beta\gamma}g^{\gamma\alpha}\rho_E^\alpha(g^{\alpha\beta}g^{\beta\gamma}g^{\gamma\alpha})^{-1} = \rho_E^\alpha$.

Property 2. *The 2-transition functions satisfy the following generalized cocycle relations:* $\forall x \in U^\alpha \cap U^\beta \cap U^\gamma$,

$$h^{\alpha\beta\gamma}(x) = g^{\alpha\beta}(x)h^{\beta\gamma\alpha}(x)g^{\alpha\beta}(x)^{-1} \quad (48)$$

$$h^{\alpha\gamma\beta}(x) = h^{\alpha\beta\gamma}(x)^{-1} \quad (49)$$

$$h^{\beta\alpha\gamma}(x) = g^{\alpha\beta}(x)^{-1}h^{\alpha\beta\gamma}(x)^{-1}g^{\alpha\beta}(x) \quad (50)$$

and $\forall x \in U^\alpha \cap U^\beta \cap U^\gamma \cap U^\delta$,

$$h^{\alpha\delta\gamma}(x)h^{\alpha\gamma\beta}(x) = h^{\alpha\delta\beta}(x)g^{\alpha\beta}(x)h^{\beta\delta\gamma}(x)g^{\alpha\beta}(x)^{-1}. \quad (51)$$

Proof.

$$h^{\alpha\beta\gamma} = g^{\alpha\beta}g^{\beta\gamma}g^{\gamma\alpha} = g^{\alpha\beta}g^{\beta\gamma}g^{\gamma\alpha}g^{\alpha\beta}g^{\beta\alpha} = g^{\alpha\beta}h^{\beta\gamma\alpha}g^{\beta\alpha} \quad (52)$$

$$h^{\beta\alpha\gamma} = g^{\beta\alpha}g^{\alpha\gamma}g^{\gamma\beta} = g^{\beta\alpha}g^{\alpha\gamma}g^{\gamma\beta}g^{\beta\alpha}g^{\alpha\beta} = (g^{\alpha\beta})^{-1}(h^{\alpha\beta\gamma})^{-1}g^{\alpha\beta} \quad (53)$$

$$h^{\alpha\gamma\beta} = g^{\alpha\gamma}g^{\gamma\beta}g^{\beta\alpha} = (g^{\alpha\beta}g^{\beta\gamma}g^{\gamma\alpha})^{-1} = (h^{\alpha\beta\gamma})^{-1}, \quad (54)$$

finally

$$g^{\alpha\delta} = h^{\alpha\delta\gamma}g^{\alpha\gamma}g^{\gamma\delta} = h^{\alpha\delta\gamma}h^{\alpha\gamma\beta}g^{\alpha\beta}g^{\beta\gamma}g^{\gamma\delta} \quad (55)$$

and

$$g^{\alpha\delta} = h^{\alpha\delta\beta}g^{\alpha\beta}g^{\beta\delta} = h^{\alpha\delta\beta}g^{\alpha\beta}h^{\beta\delta\gamma}g^{\beta\gamma}g^{\gamma\delta}; \quad (56)$$

we conclude that

$$h^{\alpha\delta\gamma}h^{\alpha\gamma\beta}g^{\alpha\beta} = h^{\alpha\delta\beta}g^{\alpha\beta}h^{\beta\delta\gamma}. \quad (57)$$

□

4. Geometric phases of open quantum systems

4.1. Generator of the C^* -geometric phases

Now we are able to define a generator of the geometric phases for open quantum systems by analogy with the geometric phases of dissipative quantum systems: $\mathcal{A}^\alpha \in \Omega^1(U^\alpha, \mathcal{L}(\mathcal{H}_S))$ is a solution of the following equation:

$$\mathcal{A}^\alpha \|\phi_E^\alpha\|_*^2 = \langle \phi_E^\alpha | d\phi_E^\alpha \rangle_* \quad (58)$$

Property 3. *The C^* -geometric phase generator and the mixed eigenstate are related by the following equation:*

$$d\rho_E^\alpha = \mathcal{A}^\alpha \rho_E^\alpha + \rho_E^\alpha (\mathcal{A}^\alpha)^\dagger \quad (59)$$

Proof.

$$\begin{aligned} d\rho_E &= \text{tr}_{\mathcal{H}_E} (|d\phi_E\rangle \langle \phi_E| + |\phi_E\rangle \langle d\phi_E|) \\ &= \langle \phi_E | d\phi_E \rangle_* + \langle \phi_E | d\phi_E \rangle_*^\dagger \\ &= \mathcal{A}\rho_E + (\mathcal{A}\rho_E)^\dagger. \end{aligned} \quad (60)$$

□

Property 4. *Under a gauge transformation $\tilde{\phi}_E^\alpha(x) = g(x)k(x)\phi_E^\alpha(x)$ with $g \in \Gamma(U^\alpha, P_G^\alpha)$ and $k \in \Gamma(U^\alpha, P_K^\alpha)$, the C^* -geometric phase generator is transformed as follows:*

$$\tilde{\mathcal{A}}^\alpha = g\mathcal{A}^\alpha g^{-1} + dg g^{-1} + g\eta g^{-1}, \quad (61)$$

with $\eta \in \Omega^1(U^\alpha, P_J^\alpha) = \Omega^1(U^\alpha) \otimes_M \Gamma(U^\alpha, P_J^\alpha)$ the solution of

$$\eta \|\phi_E^\alpha\|_*^2 = \langle \phi_E^\alpha | k^{-1} dk \phi_E^\alpha \rangle_* \quad (62)$$

$\Gamma(U^\alpha, P_G^\alpha)$ and $\Gamma(U^\alpha, P_K^\alpha)$ denote the sets of sections of P_G^α and P_K^α , respectively.

Proof.

$$\begin{aligned} \tilde{\mathcal{A}} \|\tilde{\phi}_E\|_*^2 &= \langle \tilde{\phi}_E | d\tilde{\phi}_E \rangle_* \\ &\Rightarrow \tilde{\mathcal{A}} g \|\phi_E\|_*^2 g^\dagger = \text{tr}_{\mathcal{H}_E} (k dg |\phi_E\rangle \langle \phi_E| g^\dagger k^{-1}) \\ &\quad + \text{tr}_{\mathcal{H}_E} (g dk |\phi_E\rangle \langle \phi_E| k^{-1} g^\dagger) \\ &\quad + \text{tr}_{\mathcal{H}_E} (gk |d\phi_E\rangle \langle \phi_E| k^{-1} g^\dagger) \\ &\Rightarrow \tilde{\mathcal{A}} g \|\phi_E\|_*^2 g^\dagger = dg \|\phi_E\|_*^2 g^\dagger + g\mathcal{A} \|\phi_E\|_*^2 g^\dagger + g\eta \|\phi_E\|_*^2 g^\dagger \\ &\Rightarrow g^{-1} \tilde{\mathcal{A}} g \|\phi_E\|_*^2 = (g^{-1} dg + \mathcal{A} + \eta) \|\phi_E\|_*^2 \\ &\Rightarrow \tilde{\mathcal{A}} = dg g^{-1} + g\mathcal{A} g^{-1} + g\eta g^{-1} \end{aligned} \quad (63)$$

$$\begin{aligned} \langle \phi_E | k^{-1} dk |\phi_E \rangle_*^\dagger &= \langle \phi_E | d(k^{-1} k) |\phi_E \rangle_* \\ &\Rightarrow \langle \phi_E | k^{-1} dk |\phi_E \rangle_*^\dagger = -\langle \phi_E | k^{-1} dk |\phi_E \rangle_* \\ &\Rightarrow \eta \rho_E + \rho_E \eta^\dagger = 0 \\ &\Rightarrow \eta \in \Omega^1(U^\alpha, P_{Jx}^\alpha). \end{aligned} \quad (64)$$

□

Remark. $dg g^{-1} \in \Omega^1(U^\alpha, P_g^\alpha)$. We see that the K_x -gauge transformations on the pure state of the universe ϕ_E induce j_x -gauge transformations on the C^* -geometric phase generator. G_x is the gauge group associated with the dynamics of the quantum system (the equivalent of the gauge group for closed quantum systems), whereas J_x is the gauge group associated with the decoherence induced by the environment (by its relation with K_x , the gauge group associated with the dynamics of the environment).

When ρ_E^α is invertible, equation (58) has a single solution:

$$\mathcal{A}^\alpha = \langle \phi_E^\alpha | d\phi_E^\alpha \rangle_* (\|\phi_E^\alpha\|_*^2)^{-1} \quad (65)$$

but there can be difficulty in applying this, particularly if the precise description of the universe is unknown. This problem is partially solved by the fact that we can prove (see appendix B) that the C^* -geometric phase generator is equal to $\tilde{\mathcal{A}}^\alpha = \frac{1}{2} d\rho_E^\alpha (\rho_E^\alpha)^{-1}$ modulo a j_x -gauge transformation. If ρ_E^α is not invertible, equation (58) can have several solutions, but they are equal modulo a j_x -gauge transformation. This result and other mathematical properties of the C^* -geometric phase generator can be found in appendix B.

Property 5. We assume that E is non-degenerate. The C^* -geometric phase generator can be decomposed as $\mathcal{A}^\alpha = A^\alpha + R^\alpha$ where the reduced generator $A^\alpha \in \Omega^1(U^\alpha, P_g^\alpha)$ is called the gauge potential (or the G -potential) and where the remainder R^α is almost zero in the sense $\text{tr}_{\mathcal{H}_S}(\rho_E^\alpha R^\alpha) = 0$.

Proof. Let $P_E^\alpha(x) \in \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$ be the orthogonal projector (in sense of $\langle\langle \cdot | \cdot \rangle\rangle$) on $\ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E})$ continuous with respect to $x \in U^\alpha$:

$$\begin{aligned} \mathcal{A} \|\phi_E\|_*^2 &= \langle \phi_E | d\phi_E \rangle_* \\ &= \langle \phi_E | P_E d\phi_E \rangle_* + \langle \phi_E | (1_{\mathcal{H}_S \otimes \mathcal{H}_E} - P_E) d\phi_E \rangle_*. \end{aligned} \quad (66)$$

We set then A and R as the solutions of the following equations:

$$A \|\phi_E\|_*^2 = \langle \phi_E | P_E d\phi_E \rangle_* \quad (67)$$

$$R \|\phi_E\|_*^2 = \langle \phi_E | (1 - P_E) d\phi_E \rangle_*. \quad (68)$$

Byproperty 14, we have $\langle \phi_E^\alpha | P_E^\alpha d\phi_E^\alpha \rangle_* \in \Omega^1(U^\alpha, P_g^\alpha)$ and then $A^\alpha \in \Omega^1(U^\alpha, P_g^\alpha)$:

$$\begin{aligned} \text{tr}_{\mathcal{H}_S}(\rho_E R) &= \text{tr}_{\mathcal{H}_S} \langle \phi_E | (1 - P_E) d\phi_E \rangle_* \\ &= \langle\langle \phi_E | (1 - P_E) d\phi_E \rangle\rangle \\ &= \langle\langle (1 - P_E) \phi_E | d\phi_E \rangle\rangle \\ &= 0. \end{aligned} \quad (69)$$

□

Property 6. Under a gauge transformation $\tilde{\phi}_E^\alpha(x) = g(x)k(x)\phi_E^\alpha(x)$ with $g \in \Gamma(U^\alpha, P_G^\alpha)$ and $k \in \Gamma(U^\alpha, P_K^\alpha)$, the G -potential is transformed as follows:

$$\tilde{A}^\alpha = gA^\alpha g^{-1} + dg g^{-1} + g\eta g^{-1}, \quad (70)$$

with $\eta \in \Omega^1(U^\alpha, P_j^\alpha)$ the solution of

$$\eta \|\phi_E^\alpha\|_*^2 = \langle \phi_E^\alpha | k^{-1} dk \phi_E^\alpha \rangle_* \quad (71)$$

and the remainder is transformed as follows:

$$\tilde{R}^\alpha = gR^\alpha g^{-1}. \quad (72)$$

Proof. By definition, $gP_E = P_E g$ and $kP_E = P_E k$, and since $dg g^{-1} \in \Omega^1(U^\alpha, P_g^\alpha)$ and $dk k^{-1} \in \Omega^1(U^\alpha, P_k^\alpha)$, we have $(1_{\mathcal{H}_S \otimes \mathcal{H}_E} - P_E) dg g^{-1} = 0$ and $(1_{\mathcal{H}_S \otimes \mathcal{H}_E} - P_E) dk k^{-1} = 0$. The rest of the proof consists in remarking the proof of property 4 by considering these identities. \square

We consider now the gauge transformations at the intersection of several charts. Let $g^{\alpha\beta} \in \mathcal{C}^\infty(U^\alpha \cap U^\beta) \otimes_M \bigsqcup_x G_x$ and $k^{\alpha\beta} \in \mathcal{C}^\infty(U^\alpha \cap U^\beta) \otimes_M \bigsqcup_x K_x$ be the 1-transition functions. We have clearly $\forall x \in U^\alpha \cap U^\beta$

$$A^\beta = (g^{\alpha\beta})^{-1} A^\alpha g^{\alpha\beta} - (g^{\alpha\beta})^{-1} dg^{\alpha\beta} + (g^{\alpha\beta})^{-1} \eta^{\alpha\beta} g^{\alpha\beta}, \quad (73)$$

where $\eta^{\alpha\beta} \in \Omega^1(U^\alpha \cap U^\beta) \otimes \bigsqcup_x j_x$ is a solution of the equation

$$\eta^{\alpha\beta} \|\phi_E^\alpha\|_*^2 = \langle \phi_E^\alpha | k^{\alpha\beta} d(k^{\alpha\beta})^{-1} | \phi_E^\alpha \rangle_*, \quad (74)$$

where $\eta^{\alpha\beta}$ is called the potential transformation (or the J -potential) and is associated with the decoherence induced by the environment (since this term vanishes for a closed quantum system).

Property 7. $\forall x \in U^\alpha \cap U^\beta \cap U^\gamma$, the potential transformation satisfies

$$\eta^{\alpha\beta} + g^{\alpha\beta} \eta^{\beta\gamma} (g^{\alpha\beta})^{-1} - h^{\alpha\beta\gamma} \eta^{\alpha\gamma} (h^{\alpha\beta\gamma})^{-1} = dh^{\alpha\beta\gamma} (h^{\alpha\beta\gamma})^{-1} - [A^\alpha, h^{\alpha\beta\gamma}] (h^{\alpha\beta\gamma})^{-1} \quad (75)$$

where $h^{\alpha\beta\gamma} \in \mathcal{C}^\infty(U^\alpha \cap U^\beta \cap U^\gamma) \otimes_M \bigsqcup_x J_x$ is the 2-transition function.

Proof.

$$\begin{aligned} \eta^{\alpha\beta} &= g^{\alpha\beta} A^\beta (g^{\alpha\beta})^{-1} - A^\alpha + dg^{\alpha\beta} (g^{\alpha\beta})^{-1} \\ &= g^{\alpha\beta} (g^{\beta\gamma} A^\gamma (g^{\beta\gamma})^{-1} + dg^{\beta\gamma} (g^{\beta\gamma})^{-1} - \eta^{\beta\gamma}) (g^{\alpha\beta})^{-1} - A^\alpha + dg^{\alpha\beta} (g^{\alpha\beta})^{-1} \\ &= h^{\alpha\beta\gamma} A^\alpha (h^{\alpha\beta\gamma})^{-1} - A^\alpha - g^{\alpha\beta} \eta^{\beta\gamma} (g^{\alpha\beta})^{-1} \\ &\quad \left. \begin{aligned} &- g^{\alpha\beta} g^{\beta\gamma} \eta^{\gamma\alpha} (g^{\alpha\beta} g^{\beta\gamma})^{-1} \\ &+ g^{\alpha\beta} g^{\beta\gamma} dg^{\gamma\alpha} (g^{\alpha\beta} g^{\beta\gamma} g^{\gamma\alpha})^{-1} \\ &+ g^{\alpha\beta} dg^{\beta\gamma} (g^{\alpha\beta} g^{\beta\gamma})^{-1} \\ &+ dg^{\alpha\beta} (g^{\alpha\beta})^{-1} \end{aligned} \right\} dh^{\alpha\beta\gamma} (h^{\alpha\beta\gamma})^{-1}, \quad (76) \end{aligned}$$

now

$$\begin{aligned} \eta^{\gamma\alpha} &= g^{\gamma\alpha} A^\alpha (g^{\gamma\alpha})^{-1} + dg^{\gamma\alpha} (g^{\gamma\alpha})^{-1} - A^\gamma \\ &= A^\gamma + g^{\gamma\alpha} d(g^{\gamma\alpha})^{-1} - g^{\gamma\alpha} \eta^{\alpha\gamma} (g^{\gamma\alpha})^{-1} + dg^{\gamma\alpha} (g^{\gamma\alpha})^{-1} - A^\gamma \\ &= -g^{\gamma\alpha} \eta^{\alpha\gamma} (g^{\gamma\alpha})^{-1}; \quad (77) \end{aligned}$$

then

$$\begin{aligned} \eta^{\alpha\beta} &= h^{\alpha\beta\gamma} A^\alpha (h^{\alpha\beta\gamma})^{-1} - A^\alpha - g^{\alpha\beta} \eta^{\beta\gamma} (g^{\alpha\beta})^{-1} \\ &\quad + g^{\alpha\beta} g^{\beta\gamma} g^{\gamma\alpha} \eta^{\alpha\gamma} (g^{\alpha\beta} g^{\beta\gamma} g^{\gamma\alpha})^{-1} + dh^{\alpha\beta\gamma} (h^{\alpha\beta\gamma})^{-1}. \quad (78) \end{aligned}$$

\square

4.2. The higher gauge theory associated with the C^* -geometric phase

The gauge transformations of the gauge potential A^α and of the potential transformation $\eta^{\alpha\beta}$ are characteristic of a higher gauge theory, as described with several different notations in [26–31]. In accordance with these previous works, we introduce here three types of curvature.

Let $B^\alpha = dA^\alpha - A^\alpha \wedge A^\alpha \in \Omega^2(U^\alpha, \mathcal{L}(\mathcal{H}_S))$ be the curving and $F^\alpha = dR^\alpha - [A^\alpha, R^\alpha] - R^\alpha \wedge R^\alpha \in \Omega^2(U^\alpha, \mathcal{L}(\mathcal{H}_S))$ be the fake curvature:

$$B^\alpha = dA^\alpha - A^\alpha \wedge A^\alpha + F^\alpha. \tag{79}$$

We can prove (see appendix C) that in fact $B^\alpha \in \Omega^2(U^\alpha, P_j^\alpha)$ and $F^\alpha \in \Omega^2(U^\alpha, P_g^\alpha)$. Finally we set $H^\alpha \in \Omega^3(U^\alpha, P_j^\alpha)$, the true curvature, defined by

$$\begin{aligned} H^\alpha &= dB^\alpha - [A^\alpha, B^\alpha] \\ &= dF^\alpha - [A^\alpha, F^\alpha]. \end{aligned} \tag{80}$$

These three types of curvature satisfy some gauge transformation formulae (studied in appendix C) which are also characteristic of a higher gauge theory [26–31]).

The C^* -geometric phases of an open quantum system are not related to a principal bundle over M (as would occur for the geometric phases of closed quantum systems) because the family of local principal G -bundles $\{P_G^\alpha\}_\alpha$ cannot be lifted to a single global principal bundle (this is due to the failure of the cocycle relation with $g^{\alpha\beta}$ measured by $h^{\alpha\beta\gamma}$). The geometric structure \mathcal{P} associated with the family $\{P_G^\alpha\}_\alpha$ does not define a manifold but a category (see [35] for a presentation of category theory); this feature is specific to a higher gauge theory.

The geometric structure \mathcal{P} is defined from the 1-transition $g^{\alpha\beta}$ and the 2-transition $h^{\alpha\beta\gamma}$ and is endowed with a 2-connection (see [29–31]) defined from $(A^\alpha, B^\alpha, \eta^{\alpha\beta})$. \mathcal{P} is a non-Abelian bundle gerbe [26–31])² with the structure Lie crossed module (G, J, t, Ad) where $t : J \rightarrow G$ is the canonical injection (J is a subgroup of G) and $\text{Ad} : G \rightarrow \text{Aut}(J)$ is the adjoint representation of G on itself restricted to J (in the homomorphism domain).

The replacement of the single gauge group of closed quantum systems by a gauge Lie crossed module for open quantum systems is explained by the need for a gauge group associated with the evolution of the quantum system (G) and also for a gauge group associated with the decoherence induced by the environment (J).

Equivalently, \mathcal{P} can be viewed as a principal categorical bundle (a 2-bundle, see [29–31]) on the left with the structure groupoid \mathcal{G} having $\text{Obj}(\mathcal{G}) = G$ as set of objects and $\text{Morph}(\mathcal{G}) = J \rtimes G$ as set of arrows, the semi-direct product (the horizontal composition of arrows) being defined by $(h, g)(h', g') = (h\text{Ad}(g)h', gg')$. The source map of \mathcal{G} is defined by $s(h, g) = g$ and the target map is defined by $t(h, g) = t(h)g$ (with $t(h)$ the canonical injection of h in G). The morphism composition (the vertical composition of arrows) is defined by $(h, g) \circ (h', t(h)g) = (hh', g)$.

The explicit constructions of \mathcal{P} , of its 2-connection and of its total category are somewhat technical but can be found in appendix C. In the following section, we show that at the adiabatic limit, the time-dependent mixed states exhibit C^* -geometric phases which are related to horizontal lifts in \mathcal{P} .

² In some references as [26–28], the authors consider only the special crossed module $(\text{Aut}(G), G, \text{Ad}, \text{id}_{\text{Aut}(G)})$ for a Lie group G .

5. The adiabatic transport of mixed states

The study of rigorous adiabatic theorems for mixed states is not the subject of this paper but appendix D presents a heuristic approach to the adiabatic approximation for the $*$ -eigenvectors. This approach shows that the adiabatic condition can be expressed as being $R_\mu^\alpha(x(t))\dot{x}^\mu(t) \simeq 0$ ($\forall t$) where R^α is the remainder of the C^* -geometric phase generator. This property shows the interpretation of R^α ; it measures the non-adiabatic effects. The fake curvature $F^\alpha = dR^\alpha - R^\alpha \wedge R^\alpha - [A^\alpha, R^\alpha]$ then characterizes the influence of these non-adiabatic effects on the geometry of \mathcal{P} and the curving transformation $\chi^{\alpha\beta} = [R^\alpha, \eta^{\alpha\beta}]$ measures the intertwining between the non-adiabatic effects and the decoherence induced by the environment.

We now are able to introduce the adiabatic transport of a mixed state.

Property 8. Let $(\mathcal{D}(\mathcal{H}_S), \mathcal{L}_x, M)$ be an open quantum dynamical system associated with a universe described by the quantum dynamical system $(\mathcal{H}_S \otimes \mathcal{H}_E, H(x), M)$. Let $U^\alpha \ni x \mapsto \rho_E^\alpha(x) \in \mathcal{D}(\mathcal{H}_S)$ be a mixed eigenstate of the open quantum system and $M \ni x \mapsto E(x)$ be the associated eigenoperator supposed non-degenerate. Let $t \mapsto x(t) \in M$ be an evolution corresponding to a path \mathcal{C} within U^α . Let $t \mapsto \rho(t) \in \mathcal{D}(\mathcal{H}_S)$ be the solution of the Lindblad equation $i\hbar \frac{d\rho}{dt} = \mathcal{L}_{x(t)}(\rho(t))$ with $\rho(0) = \rho_E^\alpha(x(0))$. We assume the adiabatic condition (D.1), i.e. $R_\mu^\alpha(x(t))\dot{x}^\mu(t) \simeq 0$. We can then write $\forall t > 0$

$$\rho(t) = g_{EA}(t)\rho_E(x(t))g_{EA}(t)^\dagger, \quad (81)$$

with

$$g_{EA}(t) = \mathbb{T} e^{-i\hbar^{-1} \int_0^t E(x(t')) dt'} \mathbb{P}_{\mathcal{C}} e^{-\int_{x(0)}^{x(t)} (A^\alpha(x) + \eta(x))} \in G_{x(t)}, \quad (82)$$

where $A^\alpha \in \Omega^1(U^\alpha, P_g^\alpha)$ is the gauge potential and $\eta \in \Omega^1(\mathcal{C}, P_j^\alpha)$ is a particular j_x -gauge transformation.

Proof. Let $\psi \in \mathcal{H}_S \otimes \mathcal{H}_E$ be the solution of the Schrödinger equation $i\hbar \frac{d\psi}{dt} = H(x(t))\psi(t)$ with $\psi(0) = \phi_E(x(0))$ where ϕ_E is the $*$ -eigenvector associated with E and ρ_E . By application of theorem 1, we know that $\forall t, \psi(t) \in \ker(H(x(t)) - E(x(t)))$, and since the action of $G_x \times K_x$ is transitive (E is non-degenerate), we have

$$\forall t, \quad \exists g_{EA}(t) \in G_{x(t)}, \quad \exists k_\eta(t) \in K_{x(t)}, \quad \psi(t) = k_\eta(t)g_{EA}(t)\phi_E(x(t)). \quad (83)$$

By inserting this expression into the Schrödinger equation of the universe, we find

$$\begin{aligned} i\hbar k_\eta \dot{g}_{EA} \phi_E + i\hbar \dot{k}_\eta g_{EA} \phi_E + i\hbar k_\eta g_{EA} \frac{d\phi_E}{dt} &= H k_\eta g_{EA} \phi_E \\ &= E k_\eta g_{EA} \phi_E \end{aligned} \quad (84)$$

$$g_{EA}^{-1} \dot{g}_{EA} \phi_E = -\frac{d\phi_E}{dt} - k_\eta^{-1} \dot{k}_\eta \phi_E - i\hbar^{-1} g_{EA}^{-1} E g_{EA} \phi_E \quad (85)$$

and then

$$g_{EA}^{-1} \dot{g}_{EA} |\phi_E\rangle \langle \phi_E| = -\left| \frac{d\phi_E}{dt} \right\rangle \langle \phi_E| - k_\eta^{-1} \dot{k}_\eta |\phi_E\rangle \langle \phi_E| - i\hbar^{-1} g_{EA}^{-1} E g_{EA} |\phi_E\rangle \langle \phi_E|. \quad (86)$$

By taking the partial trace on \mathcal{H}_E of this expression, we find

$$g_{EA}^{-1} \dot{g}_{EA} \rho_E = -(\mathcal{A}_\mu + \eta_\mu) \dot{x}^\mu \rho_E - i\hbar^{-1} g_{EA}^{-1} E g_{EA} \rho_E, \quad (87)$$

where the potential transformation η is a solution of the following equation:

$$\eta(x(t)) \|\phi_E(x(t))\|_*^2 = \langle \phi_E(x(t)) | k_\eta(x(t))^{-1} \frac{dk_\eta(x(t))}{dt} \phi_E(x(t)) \rangle_* dt. \quad (88)$$

Since $R_\mu \dot{x}^\mu \simeq 0$, $\mathcal{A}_\mu \dot{x}^\mu \simeq A_\mu \dot{x}^\mu$, we can then choose

$$g_{EA}^{-1} \dot{g}_{EA} = -(A_\mu + \eta_\mu) \dot{x}^\mu - i\hbar^{-1} g_{EA}^{-1} E g_{EA}. \quad (89)$$

We set

$$\begin{aligned} g_A(t) &= \mathbb{T} e^{i\hbar^{-1} \int_0^t E(t) dt} g_{EA}(t) \\ \iff g_{EA}(t) &= \mathbb{T} e^{-i\hbar^{-1} \int_0^t E(t) dt} g_A(t). \end{aligned} \quad (90)$$

By inserting this expression into the previous equation, we find

$$g_A^{-1} \dot{g}_A = -(A_\mu + \eta_\mu) \dot{x}^\mu \quad (91)$$

and thus

$$g_A = \mathbb{P}_C e^{-\int_{x(0)}^{x(t)} (A(x) + \eta(x))}. \quad (92)$$

□

We note that even if the C^* -geometric phase $\mathbb{P}_C e^{\int_{x(0)}^{x(t)} (A(x) + \eta(x))}$ is non-Abelian since it belongs to $\mathcal{GL}(\mathcal{H}_S)$, it is the equivalent of an Abelian geometric phase of a closed quantum system. The non-Abelianity results from the substitution of the Abelian ring \mathbb{C} by the non-Abelian C^* -algebra $\mathcal{L}(\mathcal{H}_S)$ (the paradigm of the present approach).

A^α being not anti-self-adjoint, g_{EA} is not unitary; nevertheless, the normalization of the density matrix trace is preserved by the adiabatic transport formula (see appendix D). If the path \mathcal{C} crosses several charts, the C^* -geometric phase is obtained by a formula similar to equation (6).

6. Illustrative example: the control of a qubit

In this section, we present a very simple example in order to illustrate the concepts introduced in this paper with a physical toy model. For the sake of simplicity, we do not consider the question of the chart indices in this section.

6.1. The model

We consider a qubit described by a CAR algebra (a fermionic algebra) acting on $\mathcal{H}_S = \mathbb{C}^2$:

$$\begin{aligned} c|0\rangle &= 0 & c^\dagger|0\rangle &= |1\rangle \\ c|1\rangle &= |0\rangle & c^\dagger|1\rangle &= 0, \end{aligned} \quad (93)$$

with $cc^\dagger + c^\dagger c = 1_{\mathcal{H}_S}$. This qubit is subjected to a decoherence process described by a phase damping model (see [37]):

$$H_{I_0} = \chi c^\dagger c \otimes (b + b^\dagger), \quad (94)$$

where $\chi \in \mathbb{R}$ is a constant. Usually $\{b, b^\dagger, 1_{\mathcal{H}_\varepsilon}\}$ constitutes a CCR algebra (a harmonic oscillator algebra), but in the present work for the sake of simplicity and to avoid some technical difficulties, we consider that it is also a CAR algebra (the environment is then constituted by a single-mode fermionic bath). We then have the following Hamiltonian:

$$H_0 = \hbar\omega_c c^\dagger c \otimes 1_{\mathcal{H}_\varepsilon} + \hbar\omega_b 1_{\mathcal{H}_S} \otimes b^\dagger b + \chi c^\dagger c \otimes (b + b^\dagger), \quad (95)$$

where $\omega_c, \omega_b \in \mathbb{R}^{+*}$ are some constants. We control the qubit by operating on it with rotations. Let $U(x) = e^{ix^\mu \sigma_\mu} \in \mathcal{U}(\mathcal{H}_S)$ be a qubit rotation, where $x = (x^0, x^1, x^3) \in \mathbb{R}^3$ are the control

parameters and $\{\sigma_\mu\}_{\mu=1,2,3}$ are the Pauli matrices. The Hamiltonian of the qubit driven by rotations is then

$$\begin{aligned} H(x) &= U(x)H_0U(x)^{-1} \\ &= \hbar\omega_c U(x)c^\dagger cU(x)^{-1} \otimes 1_{\mathcal{H}_\mathcal{E}} + \hbar\omega_b 1_{\mathcal{H}_\mathcal{S}} \otimes b^\dagger b \\ &\quad + \chi U(x)c^\dagger cU(x)^{-1} \otimes (b + b^\dagger). \end{aligned} \quad (96)$$

Let $t \mapsto x(t)$ be a path in \mathbb{R}^3 with $x(0) = (0, 0, 0)$ and let $t \mapsto \psi(t) \in \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$ be the associated wavefunction of the universe:

$$i\hbar \frac{d\psi}{dt} = U(x(t))H_0U(x(t))^{-1}\psi(t), \quad \psi(0) = \phi_\chi, \quad (97)$$

with the initial condition $\phi_\chi \in \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$ such that $\rho_\chi = \text{tr}_{\mathcal{H}_\mathcal{E}}(|\phi_\chi\rangle\langle\phi_\chi|) = |1\rangle\langle 1|$ (at time $t = 0$, the qubit is in the pure state $|1\rangle$). Let $\phi(t) = U(x(t))^{-1}\psi(t)$ be the solution of

$$i\hbar \frac{d\phi}{dt} = \left(H_0 - i\hbar U(x(t))^{-1} \frac{dU(x(t))}{dt} \right) \phi(t), \quad \phi(0) = \phi_\chi. \quad (98)$$

The density matrix representing the evolution of the qubit is then

$$\rho(t) = U(x(t)) \text{tr}_{\mathcal{H}_\mathcal{E}}(V(t)|\phi_\chi\rangle\langle\phi_\chi|V(t)^\dagger)U(x(t))^{-1}, \quad (99)$$

with

$$V(t) = \mathbb{T} e^{-i\hbar^{-1} \int_0^t (H_0 - i\hbar U(x(t'))^{-1} \partial_{t'} U(x(t')) dt') dt'} \in \mathcal{U}(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}). \quad (100)$$

We see that the dynamics of the qubit seems to exhibit a kind of geometric phase generated by $U(x)^{-1}dU(x)$. Unfortunately it is not separated from the other matrix quantities and the expression does not involve the rotated state $U(x)|1\rangle\langle 1|U(x)^{-1}$. The formalism introduced in this paper solves these problems and clearly defines this geometric phase.

6.2. The eigenoperator and the *-eigenvector

Let $E_0 = \hbar\omega_c c c^\dagger \in \mathcal{L}(\mathcal{H}_\mathcal{S})$. We have the following usual eigenvector equation:

$$(H_0 - E_0 \otimes 1_{\mathcal{H}_\mathcal{E}})\phi_\chi = \lambda\phi_\chi, \quad (101)$$

with $\lambda = \frac{1}{2}(\sqrt{4\chi^2 + \hbar^2\omega_b^2} + 2\hbar\omega_c + \hbar\omega_b)$ (one of the four eigenvalues) and

$$\phi_\chi = \frac{2\chi|10\rangle + (\sqrt{4\chi^2 + \hbar^2\omega_b^2} + \hbar\omega_b)|11\rangle}{\sqrt{4\chi^2 + (\sqrt{4\chi^2 + \hbar^2\omega_b^2} + \hbar\omega_b)^2}}. \quad (102)$$

We note that $\rho_\chi = |1\rangle\langle 1|$.

We consider the following eigenoperator and *-eigenvector of $H(x)$:

$$E(x) = U(x)(E_0 + \lambda 1_{\mathcal{H}_\mathcal{S}})U(x)^{-1}, \quad \phi_E(x) = U(x)\phi_\chi. \quad (103)$$

It is easy to verify that

$$H(x)\phi_E(x) = E(x)\phi_E(x) \text{ and } [H(x), E(x)] = 0. \quad (104)$$

The mixed eigenstate is then $\rho_E(x) = U(x)|1\rangle\langle 1|U(x)^{-1}$ which is as expected the rotated initial state. The associated gauge invariances are then described by the following Lie algebras:

$$\mathfrak{g}_x = \left\{ U(x) \begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix} U(x)^{-1}; \alpha, \gamma \in \mathbb{C}, \beta \in \mathbb{C} \right\} \quad (105)$$

$$\mathfrak{j}_x^0 = \left\{ U(x) \begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix} U(x)^{-1}; \alpha, \gamma \in \mathbb{C}, \beta \in i\mathbb{R} \right\} \quad (106)$$

$$\mathfrak{j}_x^1 = \left\{ U(x) \begin{pmatrix} \alpha & 0 \\ \gamma & 0 \end{pmatrix} U(x)^{-1}; \alpha, \gamma \in \mathbb{C} \right\} \quad (107)$$

and $\mathfrak{k}_x = i\mathbb{R}$.

6.3. The C^* -geometric phase

By definition, the generator of the C^* -geometric phase \mathcal{A} satisfies

$$\begin{aligned} \mathcal{A}\rho_E &= \langle \phi_E | d\phi_E \rangle_* \\ &\iff \mathcal{A}U(x)|1\rangle\langle 1|U(x)^{-1} = dU(x)|1\rangle\langle 1|U(x)^{-1} \\ &\iff U(x)^{-1}\mathcal{A}U(x)|1\rangle\langle 1| = U(x)^{-1}dU(x)|1\rangle\langle 1|. \end{aligned} \quad (108)$$

This induces that

$$\mathcal{A}(x) = U(x) \begin{pmatrix} 0 & \langle 0|U(x)^{-1}dU(x)|1\rangle \\ 0 & \langle 1|U(x)^{-1}dU(x)|1\rangle \end{pmatrix} U(x)^{-1} \quad (109)$$

modulo a j_x^1 -gauge transformation. \mathcal{A} exhibits the relevant components of the generator $U^{-1}(x)dU(x)$ postulated at the beginning of this section. Moreover, we have

$$\begin{aligned} \langle \phi_E | P_E d\phi_E \rangle_* &= \langle \langle \phi_\chi | U(x)^{-1}dU(x)|\phi_\chi \rangle \rangle U(x)|1\rangle\langle 1|U(x)^{-1} \\ &= \langle 1|U(x)^{-1}dU(x)|1\rangle U(x)|1\rangle\langle 1|U(x)^{-1}, \end{aligned} \quad (110)$$

where $P_E = |\phi_E\rangle\langle \phi_E|$. This shows that the reduced geometric phase generator is

$$A(x) = U(x) \begin{pmatrix} 0 & 0 \\ 0 & \langle 1|U(x)^{-1}dU(x)|1\rangle \end{pmatrix} U(x)^{-1}. \quad (111)$$

The adiabatic condition $R_\mu(x(t))\dot{x}^\mu(t) \simeq 0$ is then equivalent to

$$\langle 0|U(x)^{-1} \frac{\partial U(x)}{\partial x^\mu} |1\rangle \dot{x}^\mu(t) \simeq 0. \quad (112)$$

The non-adiabatic coupling between $|1\rangle$ and $|0\rangle$ induced by the rotation must then be negligible; this is in accordance with the intuitive significance of the adiabatic approximation.

Finally, by applying property 8, we have

$$\rho(t) = g_{EA}(t)U(x(t))|1\rangle\langle 1|U(x(t))^{-1}g_{EA}(t)^{-1}, \quad (113)$$

with

$$g_{EA}(t) = \mathbb{T} e^{-i\hbar^{-1} \int_0^t U(x(t'))(E_0 + \lambda)U(x(t'))^{-1} dt'} \mathbb{P}_C e^{-\int_{x(0)}^{x(t)} \langle 1|U(x)^{-1}dU(x)|1\rangle U(x)|1\rangle\langle 1|U(x)^{-1}}. \quad (114)$$

The true geometric phase associated with the adiabatic rotation of the qubit subjected to the decoherence process explicitly appears in this formula.

7. Conclusion

The analogy between dissipative and open quantum systems, summarized by table 1, induces a new approach to extend the geometric phase concept to open quantum systems. This new kind of geometric phase defines a higher gauge theory (and not the usual gauge theory) based on the Lie crossed module (G, J, t, Ad) , where the group G is the analogue of the usual gauge group of closed quantum systems, whereas the group J is related to the decoherence induced by the environment on the open quantum system (J is reduced to $\{1_G\}$ for a closed system). The C^* -geometric phases then appear in a categorical generalization of a principal bundle defined by 1-transition functions $g^{\alpha\beta}$ associated with G but for which the cocycle relation fails, the failure being measured by 2-transition functions $h^{\alpha\beta\gamma}$ associated with J . We therefore claim that there is an equivalence between the following three phenomena.

- *Dynamics*: decoherence induced by the environment.
- *Geometry*: obstruction to lift the gauge structure onto a principal bundle.
- *Topology*: failure of the cocycle relation.

Table 1. Analogy between dissipative and open quantum systems.

Dissipative quantum systems	Open quantum systems
Ring \mathbb{C}	C^* -algebra $\mathcal{L}(\mathcal{H}_S)$
Hilbert space $\mathbb{C} \otimes \mathcal{H} = \mathcal{H}$	C^* -module $\mathcal{H}_S \otimes \mathcal{H}_E$
Dissipation density $\ \psi\ ^2 \in \mathbb{R}^{*+}$	Density matrix $\ \psi\ _*^2 \in \mathcal{D}(\mathcal{H}_S)$
$i\hbar \frac{d\ \psi\ ^2}{dt} = \mathcal{L}(\ \psi\ ^2)$	$i\hbar \frac{d\ \psi\ _*^2}{dt} = \mathcal{L}(\ \psi\ _*^2)$
$\mathcal{L}(\ \psi\ ^2) = \langle \psi (H - H^\dagger) \psi \rangle$	$\mathcal{L}(\ \psi\ _*^2) = \langle \psi H \psi \rangle_* - \langle H \psi \psi \rangle_*$
$H\phi_e = e\phi$ with $e \in \mathbb{C}$	$H\phi_E = E\phi_E$ with $E \in \mathcal{L}(\mathcal{H}_S)$
$\mathcal{L}(\ \phi_e\ ^2) = 2 \operatorname{Im}(e) \ \phi_e\ ^2$	$\mathcal{L}(\ \phi_E\ _*^2) = E \ \phi_E\ _*^2 - \ \phi_E\ _*^2 E^\dagger$
$A = \frac{\langle \phi_e d\phi_e \rangle}{\ \phi_e\ ^2}$	$\mathcal{A} \ \phi_E\ _*^2 = \langle \phi_E d\phi_E \rangle_*$
$\frac{d\ \phi_e\ ^2}{\ \phi_e\ ^2} = 2 \operatorname{Re}(A)$	$d\ \phi_E\ _*^2 = \mathcal{A} \ \phi_E\ _*^2 + \ \phi_E\ _*^2 \mathcal{A}^\dagger$

Table 2. Connection data of the higher gauge theory associated with the C^* -geometric phases.

n -form	Symbol	Degree n	Valued algebra	Interpretation
Gauge potential	A^α	1	\mathfrak{g}_x	Characterizes the adiabatic dynamics
Remainder	R^α	1	$\mathcal{L}(\mathcal{H}_S)$	Measures the non-adiabatic effects
Potential transformation	$\eta^{\alpha\beta}$	1	\mathfrak{j}_x	Manifestation of the decoherence
Curving	B^α	2	\mathfrak{j}_x	Characterizes the decoherence geometry
Fake curvature	F^α	2	\mathfrak{g}_x	Characterizes the non-adiabatic geometry
Curving transformation	$\chi^{\alpha\beta}$	2	\mathfrak{j}_x	Non-adiabaticity-decoherence intertwining
True curvature	H^α	3	\mathfrak{j}_x	Characterizes the geometry

The categorical bundle is endowed with a connection described by several data summarized in table 2 including the generator of the C^* -geometric phases.

The relation between decoherence and a higher gauge theory seems to be important. In previous works, we have already pointed out that a higher gauge theory arises for some geometric phases: in [8], for a dissipative quantum system presenting a splitting resonance crossing (the decoherence is there induced by the interplay of the dissipative process and the non-adiabatic transitions induced by the crossing), and in [32], for an atom or a molecule interacting with a modulated chirped laser field or with an irregular train of ultrashort pulses (the decoherence is there due to the photon exchanges between the quantum system and the laser field or to the chaotic behaviour of the kicked quantum system). In these previous works, the quantum systems are described by pure states, and the associated higher gauge theories are Abelian. In this work, the open quantum systems are described by mixed states and the higher gauge theory is non-Abelian.

The usual gauge theories were initially introduced to describe particle interactions (electromagnetic, electroweak or chromodynamics) and were later used to describe the geometric phases of closed quantum systems. The higher gauge theories used in the present work to describe the geometric phases of open quantum systems were initially introduced to describe string interactions. We note the interesting epistemological remark that, from the viewpoint of the gauge theories, the reduction to a more fundamental theory in high-energy physics (the passage from the quantum field theory to the string theory) seems to be equivalent in non-relativistic quantum dynamics to an increase of the complexity (the passage from a closed quantum system to a system submitted to a decoherence process).

The C^* -geometric phases presented in this paper are associated with horizontal lifts of curves in the categorical bundle. However, we can note that the higher gauge theories also permit us to define horizontal lifts of surfaces. The role of such horizontal lifts of surfaces in

the context of open quantum systems is at the present time unclear. Moreover, for some cases, J can be reduced to $U(1)$. In these cases, the generalized cocycle relation $h^{\alpha\beta\gamma}h^{\alpha\gamma\beta} = h^{\alpha\delta\beta}h^{\beta\delta\gamma}$ induces the existence of a topological invariant, the Dixmier–Douady class which measures the non-triviality of the categorical bundle. The physical interpretation of this class is also at the present time unclear.

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Appendix A. Mathematical properties of the eigenoperators

Property 9. *The mixed eigenstate $\rho_E^\alpha(x) = \|\phi_E^\alpha(x)\|_*^2$ satisfies the following equations:*

$$\mathcal{L}_x(\rho_E^\alpha(x)) = E(x)\rho_E^\alpha(x) - \rho_E^\alpha(x)E(x)^\dagger \quad (\text{A.1})$$

$$\mathcal{L}_x(E(x)\rho_E^\alpha(x) - \rho_E^\alpha(x)E(x)^\dagger) = E(x)\mathcal{L}_x(\rho_E^\alpha(x)) - \mathcal{L}_x(\rho_E^\alpha(x))E(x)^\dagger. \quad (\text{A.2})$$

Proof.

$$\begin{aligned} \mathcal{L}(\rho_E) &= \text{tr}_{\mathcal{H}_E}([H, |\phi_E\rangle\langle\phi_E|]) \\ &= \text{tr}_{\mathcal{H}_E}(|H\phi_E\rangle\langle\phi_E| - |\phi_E\rangle\langle H^\dagger\phi_E|) \\ &= E\rho_E - \rho_E E^\dagger, \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \mathcal{L}(E\rho_E - \rho_E E^\dagger) &= \text{tr}_{\mathcal{H}_E}([H, E|\phi_E\rangle\langle\phi_E|] - [H, |\phi_E\rangle\langle\phi_E|E^\dagger]) \\ &= \text{tr}_{\mathcal{H}_E}(EH|\phi_E\rangle\langle\phi_E| - E|\phi_E\rangle\langle\phi_E|H \\ &\quad - H|\phi_E\rangle\langle\phi_E|E^\dagger + |\phi_E\rangle\langle\phi_E|HE^\dagger) \\ &= E\mathcal{L}(\rho_E) - \mathcal{L}(\rho_E)E^\dagger. \end{aligned} \quad (\text{A.4})$$

□

This property can be used to find the eigenoperator and the mixed eigenstate if the Lindbladian is known without the knowledge of the precise description of the universe.

Property 10. *E is almost self-adjoint in the following sense:*

$$\forall \alpha, \forall x \in U^\alpha \quad \text{tr}_{\mathcal{H}_S}(\rho_E^\alpha(x)(E(x) - E(x)^\dagger)) = 0. \quad (\text{A.5})$$

Proof.

$$\begin{aligned} \text{tr}_{\mathcal{H}_S}(\rho_E(E - E^\dagger)) &= \text{tr}_{\mathcal{H}_S}(\rho_E E - E^\dagger \rho_E) \\ &= \text{tr}_{\mathcal{H}_S}(\mathcal{L}(\rho_E)) \\ &= \text{tr}_{\mathcal{H}_S} \text{tr}_{\mathcal{H}_E}([H, |\phi_E\rangle\langle\phi_E|]) \\ &= 0. \end{aligned} \quad (\text{A.6})$$

□

The following proposition permits us to find easily the eigenoperators and *-eigenvectors.

Proposition 1. For all $E_0(x) \in \mathcal{L}(\mathcal{H}_S)$ such that $[E_0(x) \otimes 1_{\mathcal{H}_E}, H(x)] = 0$, we have

$$H(x)\phi_{E_0,\lambda}^\alpha(x) = (E_0(x) + \lambda(x)1_{\mathcal{H}_S})\phi_{E_0,\lambda}^\alpha, \quad (\text{A.7})$$

where $\lambda \in \mathbb{C}$ and $\phi_{E_0,\lambda}^\alpha \in \mathcal{H}_S \otimes \mathcal{H}_E$ are solutions of the usual eigenvalue equation

$$(H(x) - E_0(x) \otimes 1_{\mathcal{H}_E})\phi_{E_0,\lambda}^\alpha(x) = \lambda(x)\phi_{E_0,\lambda}^\alpha(x). \quad (\text{A.8})$$

We can solve the eigenoperator equation by finding the subalgebra of $\mathcal{L}(\mathcal{H}_S)$ of the operators commuting with H , and for each element E_0 , by diagonalizing $H - E_0 \otimes 1_{\mathcal{H}_E}$.

Property 11. J_x^0 (the orbit of the isotropy subgroup of ρ_E^α by G_x) is a normal subgroup of G_x .

Proof. Let $j \in J_x^0$ and $g \in G_x$. By definition of J_x^0 , $\exists g' \in G_x$ such that $jj' \rho_E(g')^\dagger j^\dagger = g' \rho_E(g')^\dagger$. We then have

$$jg j^{-1} (g' \rho_E(g')^\dagger g^\dagger) (g^\dagger)^{-1} j^\dagger g^\dagger = g g' \rho_E(g')^\dagger g^\dagger. \quad (\text{A.9})$$

Then $jg j^{-1} \in g g' G_{x,\rho_E}(g')^\dagger g^\dagger \subset J_x^0$. We then have $g J_x^0 g^{-1} \subset J_x^0$, where J_x^0 is normal. \square

Property 12. \mathfrak{j}_x^1 is a solvable subalgebra of \mathfrak{g}_x .

Proof. $\forall O_1, O_2 \in \mathfrak{j}_x^1$,

$$\begin{aligned} [\widehat{O_1, O_2}] &= \begin{pmatrix} 0 & *_{11} \\ 0 & *_{12} \end{pmatrix} \begin{pmatrix} 0 & *_{21} \\ 0 & *_{22} \end{pmatrix} - \begin{pmatrix} 0 & *_{21} \\ 0 & *_{22} \end{pmatrix} \begin{pmatrix} 0 & *_{11} \\ 0 & *_{12} \end{pmatrix} \\ &= \begin{pmatrix} 0 & *_{11} *_{22} - *_{21} *_{12} \\ 0 & 0 \end{pmatrix}; \end{aligned} \quad (\text{A.10})$$

then

$$(\mathfrak{j}_x^1)' = \left\{ O \in \mathfrak{g}_x, \hat{O} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}. \quad (\text{A.11})$$

$\forall O_1, O_2 \in (\mathfrak{j}_x^1)'$,

$$\begin{aligned} [\widehat{O_1, O_2}] &= \begin{pmatrix} 0 & *_{11} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & *_{22} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & *_{22} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & *_{11} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \end{aligned} \quad (\text{A.12})$$

then $(\mathfrak{j}_x^1)'' = \{0\}$. \square

Property 13. $E(x) \in \mathfrak{g}_x$.

Proof.

$$HE\phi_E = EH\phi_E = EE\phi_E \Rightarrow E\phi_E \in \ker(H - E) \Rightarrow E \in \mathfrak{g}_x. \quad (\text{A.13})$$

\square

Property 14. We assume that E is non-degenerate. Let $U^\alpha \ni x \mapsto \psi^\alpha(x) \in \ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E})$ and $U^\alpha \ni x \mapsto \phi^\alpha(x) \in \ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E})$ be two continuous $*$ -eigenvectors; then $\langle \phi^\alpha | \psi^\alpha \rangle_* \in \Gamma(U^\alpha, P_{\mathfrak{g}}^\alpha)$ ($\Gamma(U^\alpha, P_{\mathfrak{g}}^\alpha)$ denotes the set of the sections of $P_{\mathfrak{g}}^\alpha$).

Proof. Since the action of $G_x \times K_x$ is transitive, we have $\mathcal{GL}(\ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E})) = G_x \times K_x$. This induces that $\mathcal{L}(\ker(H(x) - E(x) \otimes 1_{\mathcal{H}_E})) = \text{Env}(\mathfrak{g}_x \oplus \mathfrak{k}_x)$ ($\text{Env}(\mathfrak{a})$ denotes the universal enveloping algebra of \mathfrak{a}). \mathfrak{g}_x and \mathfrak{k}_x are the Lie algebras of operators which are stable by operator composition; we then have $\text{Env}(\mathfrak{g}_x \oplus \mathfrak{k}_x) = \mathbb{C} \oplus \mathfrak{g}_x \oplus \mathfrak{k}_x \oplus \mathfrak{g}_x \otimes \mathfrak{k}_x$.

$\exists X_i \in \mathfrak{g}_x$ and $\exists Y_i \in \mathfrak{k}_x$ such that

$$\begin{aligned} |\psi\rangle\langle\phi| &= \sum_i X_i \otimes Y_i \\ \Rightarrow \langle\phi|\psi\rangle_* &= \sum_i \text{tr}_{\mathcal{H}_E}(Y_i)X_i \in \mathfrak{g}_x. \end{aligned} \tag{A.14}$$

□

Appendix B. Mathematical properties of the C^* -geometric phase generator

Property 15. *The C^* -geometric phase generator is almost anti-self-adjoint in the following sense:*

$$\forall \alpha, \forall x \in U^\alpha, \quad \text{tr}_{\mathcal{H}_S}(\rho_E^\alpha(x)(\mathcal{A}(x) + \mathcal{A}(x)^\dagger)) = 0. \tag{B.1}$$

Proof.

$$\begin{aligned} \text{tr}_{\mathcal{H}_S}(\rho_E \mathcal{A}) &= \text{tr}_{\mathcal{H}_S}(\mathcal{A} \rho_E) \\ &= \text{tr}_{\mathcal{H}_S}(\langle\phi_E|\mathbf{d}\phi_E\rangle_*) \\ &= \langle\phi_E|\mathbf{d}\phi_E\rangle \\ &= -\langle\mathbf{d}\phi_E|\phi_E\rangle \\ &= -\text{tr}_{\mathcal{H}_S}(\langle\phi_E|\mathbf{d}\phi_E\rangle_*^\dagger) \\ &= -\text{tr}_{\mathcal{H}_S}(\rho_E \mathcal{A}^\dagger). \end{aligned} \tag{B.2}$$

□

Property 16. *Assuming that ρ_E^α is invertible, the C^* -geometric phase generator is*

$$\check{\mathcal{A}}^\alpha = \frac{1}{2} \mathbf{d}\rho_E^\alpha (\rho_E^\alpha)^{-1} \tag{B.3}$$

modulo a j_x -gauge transformation.

Proof.

$$\begin{aligned} \mathbf{d}\rho_E &= \frac{1}{2}(\mathbf{d}\rho_E \rho_E^{-1} \rho_E + \rho_E \rho_E^{-1} \mathbf{d}\rho_E) \\ &= \frac{1}{2}(\mathbf{d}\rho_E \rho_E^{-1} \rho_E + \rho_E (\mathbf{d}\rho_E \rho_E^{-1})^\dagger). \end{aligned} \tag{B.4}$$

$\check{\mathcal{A}}$ is then a solution of equation (59) like $\mathcal{A} = \langle\phi_E|\mathbf{d}\phi_E\rangle_* \|\phi_E\|_*^{-2}$. We then have

$$(\mathcal{A} - \check{\mathcal{A}})\rho_E + \rho_E(\mathcal{A} - \check{\mathcal{A}})^\dagger = 0 \tag{B.5}$$

and then $(\mathcal{A} - \check{\mathcal{A}}) \in \Omega^1(U^\alpha, P_{j_x}^\alpha)$. □

If ρ_E^α is not invertible, equation (58) can have several solutions, but we have the following result.

Property 17. *Assuming that $\ker \rho_E^\alpha \neq \{0\}$, if equation (58) has some solutions, then all these solutions are equal modulo j_x -gauge transformations.*

Proof. The matrix representation of $\|\phi_E\|_*^2$ in its diagonalization basis is

$$\widehat{\|\phi_E\|_*^2} = \begin{pmatrix} \varphi_{p \times p} & 0_{p \times (n-p)} \\ 0_{(n-p) \times p} & 0_{(n-p) \times (n-p)} \end{pmatrix}, \quad (\text{B.6})$$

where $\varphi_{p \times p}$ is a diagonal matrix of order p . Thus, we have

$$\hat{\mathcal{A}} \begin{pmatrix} \varphi_{p \times p} & 0_{p \times (n-p)} \\ 0_{(n-p) \times p} & 0_{(n-p) \times (n-p)} \end{pmatrix} = \langle \widehat{\phi_E} | d\phi_E \rangle_*. \quad (\text{B.7})$$

Equation (58) has some solutions if and only if in the diagonalization basis of ρ_E , the operator $\langle \phi_E | d\phi_E \rangle_*$ is represented by a matrix of the form $\begin{pmatrix} X_{p \times p} & 0_{p \times (n-p)} \\ Y_{(n-p) \times p} & 0_{(n-p) \times (n-p)} \end{pmatrix}$. The solutions are then

$$\hat{\mathcal{A}} = \begin{pmatrix} X_{p \times p} \varphi_{p \times p}^{-1} & C_{p \times (n-p)} \\ Y_{(n-p) \times p} \varphi_{p \times p}^{-1} & C_{(n-p) \times (n-p)} \end{pmatrix}, \quad (\text{B.8})$$

where $\hat{C} = \begin{pmatrix} 0_{p \times p} & C_{p \times (n-p)} \\ 0_{(n-p) \times p} & C_{(n-p) \times (n-p)} \end{pmatrix}$ is an arbitrary matrix. By construction, the associated operator C belongs to \mathfrak{j}_x . \square

We note an important difference between the two previous properties; in the first one, the gauge transformation belongs to \mathfrak{j}_x^0 , whereas in the second one, it belongs to \mathfrak{j}_x^1 .

Property 18. Assuming that ρ_E^α is invertible, and let $\check{\mathcal{A}}^\alpha = \frac{1}{2} d\rho_E^\alpha (\rho_E^\alpha)^{-1}$, then under a gauge transformation $\tilde{\rho}_E^\alpha = g\rho_E^\alpha g^\dagger$ with $g \in \Gamma(U^\alpha, P_G^\alpha)$, we have

$$\check{\mathcal{A}}^\alpha = g\check{\mathcal{A}}^\alpha g^{-1} + dg g^{-1} + g\check{\eta} g^{-1}, \quad (\text{B.9})$$

with

$$\check{\eta} = \frac{1}{2} (\rho_E^\alpha dg^\dagger (g^\dagger)^{-1} (\rho_E^\alpha)^{-1} - g^{-1} dg) \in \Omega^1(U^\alpha, P_j^\alpha). \quad (\text{B.10})$$

Proof.

$$d\tilde{\rho}_E = dg\rho_E g^\dagger + g d\rho_E g^\dagger + g\rho_E dg^\dagger \quad (\text{B.11})$$

$$\begin{aligned} \check{\mathcal{A}} &= \frac{1}{2} d\tilde{\rho}_E \tilde{\rho}_E^{-1} \\ &= \frac{1}{2} (dgg^{-1} + g d\rho_E \rho_E^{-1} g^{-1} + g\rho_E dg^\dagger (g^\dagger)^{-1} \rho_E^{-1} g^{-1}) \\ &= dgg^{-1} + g\check{\mathcal{A}} g^{-1} + \frac{1}{2} g(\rho_E dg^\dagger (g^\dagger)^{-1} \rho_E^{-1} - g^{-1} dg) g^{-1} \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} \check{\eta} \rho_E + \rho_E \check{\eta}^\dagger &= \frac{1}{2} (\rho_E dg^\dagger (g^\dagger)^{-1} - g^{-1} dg \rho_E + g^{-1} dg \rho_E - \rho_E dg^\dagger (g^\dagger)^{-1}) \\ &= 0; \end{aligned} \quad (\text{B.13})$$

then $\check{\eta} \in \Omega^1(U^\alpha, P_{j_x}^\alpha)$. \square

Appendix C. The categorical bundle and its connective structure

C1. Curving, fake curvature and true curvature

Let $B^\alpha = d\mathcal{A}^\alpha - \mathcal{A}^\alpha \wedge \mathcal{A}^\alpha \in \Omega^2(U^\alpha, \mathcal{L}(\mathcal{H}_S))$ be the curving and $F^\alpha = dR^\alpha - [A^\alpha, R^\alpha] - R^\alpha \wedge R^\alpha \in \Omega^2(U^\alpha, \mathcal{L}(\mathcal{H}_S))$ be the fake curvature:

$$B^\alpha = d\mathcal{A}^\alpha - A^\alpha \wedge \mathcal{A}^\alpha + F^\alpha. \quad (\text{C.1})$$

Property 19. $B^\alpha \in \Omega^2(U^\alpha, P_j^\alpha)$ and $F^\alpha \in \Omega^2(U^\alpha, P_g^\alpha)$.

Proof. By differentiating equation (58), we find

$$\begin{aligned} d\mathcal{A}^\alpha \|\phi_E^\alpha\|_*^2 - \mathcal{A}^\alpha \wedge d\|\phi_E^\alpha\|_*^2 &= -\text{tr}_{\mathcal{H}_E} (|d\phi_E^\alpha\rangle \wedge \langle d\phi_E^\alpha|) \\ &\Rightarrow d\mathcal{A}^\alpha \|\phi_E^\alpha\|_*^2 - \mathcal{A}^\alpha \wedge \|\phi_E^\alpha\|_*^2 - \mathcal{A}^\alpha \wedge \|\phi_E^\alpha\|_*^2 (\mathcal{A}^\alpha)^\dagger \\ &= -\text{tr}_{\mathcal{H}_E} (|d\phi_E^\alpha\rangle \wedge \langle d\phi_E^\alpha|) \\ &\Rightarrow B^\alpha \|\phi_E^\alpha\|_*^2 = \mathcal{A}^\alpha \wedge \|\phi_E^\alpha\|_*^2 (\mathcal{A}^\alpha)^\dagger - \text{tr}_{\mathcal{H}_E} (|d\phi_E^\alpha\rangle \wedge \langle d\phi_E^\alpha|). \end{aligned} \quad (\text{C.2})$$

Taking the transconjugation of this last expression, we find

$$\begin{aligned} \|\phi_E^\alpha\|_*^2 (B^\alpha)^\dagger &= (\mathcal{A}_\mu^\alpha \|\phi_E^\alpha\|_*^2 (\mathcal{A}_\nu^\alpha)^\dagger)^\dagger dx^\mu \wedge dx^\nu - \text{tr}_{\mathcal{H}_E} (|\partial_\mu \phi_E^\alpha\rangle \langle \partial_\nu \phi_E^\alpha|)^\dagger dx^\mu \wedge dx^\nu \\ &= \mathcal{A}_\nu^\alpha \|\phi_E^\alpha\|_*^2 (\mathcal{A}_\mu^\alpha)^\dagger dx^\mu \wedge dx^\nu - \text{tr}_{\mathcal{H}_E} (|\partial_\nu \phi_E^\alpha\rangle \langle \partial_\mu \phi_E^\alpha|) dx^\mu \wedge dx^\nu \\ &= -\mathcal{A}^\alpha \wedge \|\phi_E^\alpha\|_*^2 \mathcal{A}^\alpha + \text{tr}_{\mathcal{H}_E} (|d\phi_E^\alpha\rangle \wedge \langle d\phi_E^\alpha|). \end{aligned} \quad (\text{C.3})$$

Then, we have

$$\begin{aligned} B^\alpha \|\phi_E^\alpha\|_*^2 + \|\phi_E^\alpha\|_*^2 (B^\alpha)^\dagger &= 0 \\ &\Rightarrow B^\alpha \in \Omega^2(U^\alpha, P_j^\alpha). \end{aligned} \quad (\text{C.4})$$

By definition, $F^\alpha = B^\alpha - d\mathcal{A}^\alpha + \mathcal{A}^\alpha \wedge \mathcal{A}^\alpha$, $A^\alpha \in \Omega^1(U^\alpha, P_g^\alpha)$ and $B^\alpha \in \Omega^2(U^\alpha, P_j^\alpha) \subset \Omega^2(U^\alpha, P_g^\alpha)$; then $F^\alpha \in \Omega^2(U^\alpha, P_g^\alpha)$. \square

If ρ_E^α is invertible, the curving associated with \check{A}^α is $\check{B}^\alpha = \frac{1}{4} d\rho_E^\alpha (\rho_E^\alpha)^{-1} \wedge d\rho_E^\alpha (\rho_E^\alpha)^{-1}$.

Finally we set $H^\alpha \in \Omega^3(U^\alpha, P_j^\alpha)$, the true curvature defined by

$$\begin{aligned} H^\alpha &= dB^\alpha - [A^\alpha, B^\alpha] \\ &= dF^\alpha - [A^\alpha, F^\alpha]. \end{aligned} \quad (\text{C.5})$$

Let us now consider the gauge transformations at the intersection of several charts.

Property 20. $\forall x \in U^\alpha \cap U^\beta$, the curving satisfies

$$B^\beta = (g^{\alpha\beta})^{-1} B^\alpha g^{\alpha\beta} + (g^{\alpha\beta})^{-1} (d\eta^{\alpha\beta} - \eta^{\alpha\beta} \wedge \eta^{\alpha\beta} - [A^\alpha, \eta^{\alpha\beta}] + \chi^{\alpha\beta}) g^{\alpha\beta}, \quad (\text{C.6})$$

with $\chi^{\alpha\beta} = [R^\alpha, \eta^{\alpha\beta}] \in \Omega^2(U^\alpha \cap U^\beta) \otimes_M \bigsqcup_x j_x$ called the curving transformation.

Proof. From the gauge transformation formula of the C^* -geometric phase generator, we have

$$\eta^{\alpha\beta} = g^{\alpha\beta} \mathcal{A}^\beta (g^{\alpha\beta})^{-1} - \mathcal{A}^\alpha + dg^{\alpha\beta} (g^{\alpha\beta})^{-1}; \quad (\text{C.7})$$

then

$$\begin{aligned}
d\eta^{\alpha\beta} - \eta^{\alpha\beta} \wedge \eta^{\alpha\beta} &= g^{\alpha\beta} B^\beta (g^{\alpha\beta})^{-1} - B^\alpha - 2\mathcal{A}^\alpha \wedge \mathcal{A}^\alpha \\
&\quad + \mathcal{A}^\alpha \wedge g^{\alpha\beta} \mathcal{A}^\beta (g^{\alpha\beta})^{-1} + \mathcal{A}^\alpha \wedge dg^{\alpha\beta} (g^{\alpha\beta})^{-1} \\
&\quad + g^{\alpha\beta} (g^{\alpha\beta})^{-1} \wedge \mathcal{A}^\alpha + g^{\alpha\beta} \mathcal{A}^\beta (g^{\alpha\beta})^{-1} \wedge \mathcal{A}^\alpha \\
&= g^{\alpha\beta} B^\beta (g^{\alpha\beta})^{-1} - B^\alpha + \mathcal{A}^\alpha \wedge \eta^{\alpha\beta} + \eta^{\alpha\beta} \wedge \mathcal{A}^\alpha. \tag{C.8}
\end{aligned}$$

□

Property 21. $\forall x \in U^\alpha \cap U^\beta$, the fake curvature satisfies

$$F^\beta = (g^{\alpha\beta})^{-1} F^\alpha g^{\alpha\beta} - (g^{\alpha\beta})^{-1} \chi^{\alpha\beta} g^{\alpha\beta}. \tag{C.9}$$

Proof.

$$\begin{aligned}
F^\beta &= d((g^{\alpha\beta})^{-1} R^\alpha g^{\alpha\beta}) - [(g^{\alpha\beta})^{-1} A^\alpha g^{\alpha\beta} + d(g^{\alpha\beta})^{-1} g^{\alpha\beta} + (g^{\alpha\beta})^{-1} \eta^{\alpha\beta} g^{\alpha\beta}, \\
&\quad (g^{\alpha\beta})^{-1} R^\alpha g^{\alpha\beta}] - (g^{\alpha\beta})^{-1} R^\alpha \wedge R^\alpha g^{\alpha\beta} \\
&= (g^{\alpha\beta})^{-1} F^\alpha g^{\alpha\beta} - (g^{\alpha\beta})^{-1} \chi^{\alpha\beta} g^{\alpha\beta}. \tag{C.10}
\end{aligned}$$

□

Property 22. $\forall x \in U^\alpha \cap U^\beta \cap U^\gamma$, the curving transformation satisfies

$$\chi^{\alpha\beta} + g^{\alpha\beta} \chi^{\beta\gamma} (g^{\alpha\beta})^{-1} - h^{\alpha\beta\gamma} \chi^{\alpha\gamma} (h^{\alpha\beta\gamma})^{-1} = -h^{\alpha\beta\gamma} [F^\alpha, (h^{\alpha\beta\gamma})^{-1}]. \tag{C.11}$$

Proof.

$$\begin{aligned}
\chi^{\alpha\beta} &= F^\alpha - g^{\alpha\beta} F^\beta (g^{\alpha\beta})^{-1} \\
&= F^\alpha - g^{\alpha\beta} (\chi^{\beta\gamma} + g^{\beta\gamma} F^\gamma (g^{\beta\gamma})^{-1}) (g^{\alpha\beta})^{-1} \\
&= F^\alpha - g^{\alpha\beta} \chi^{\beta\gamma} (g^{\alpha\beta})^{-1} - g^{\alpha\beta} g^{\beta\gamma} \chi^{\gamma\alpha} (g^{\beta\gamma})^{-1} (g^{\alpha\beta})^{-1} - h^{\alpha\beta\gamma} F^\alpha (h^{\alpha\beta\gamma})^{-1} \tag{C.12}
\end{aligned}$$

but

$$\begin{aligned}
\chi^{\gamma\alpha} &= F^\gamma - g^{\gamma\alpha} F^\alpha (g^{\gamma\alpha})^{-1} \\
&= g^{\gamma\alpha} (g^{\alpha\gamma} F^\gamma (g^{\alpha\gamma})^{-1} - F^\alpha) (g^{\gamma\alpha})^{-1} \\
&= -g^{\gamma\alpha} \chi^{\alpha\gamma} (g^{\gamma\alpha})^{-1}; \tag{C.13}
\end{aligned}$$

then

$$\chi^{\alpha\beta} = F^\alpha - g^{\alpha\beta} \chi^{\beta\gamma} (g^{\alpha\beta})^{-1} + h^{\alpha\beta\gamma} \chi^{\alpha\gamma} (h^{\alpha\beta\gamma})^{-1} - h^{\alpha\beta\gamma} F^\alpha (h^{\alpha\beta\gamma})^{-1}. \tag{C.14}$$

□

C.2. The fibred structure

C.2.1. Construction of \mathcal{P} as a non-Abelian bundle gerbe. Let $\xi^{\alpha\beta} : P_{G|U^\alpha \cap U^\beta}^\beta \rightarrow P_{G|U^\alpha \cap U^\beta}^\alpha$ be the bundle isomorphism defined by

$$\forall p \in P_{G|U^\alpha \cap U^\beta}^\beta, \quad \xi^{\alpha\beta}(p) = g^{\alpha\beta}(\pi_{P_G^\beta}(p))p, \tag{C.15}$$

where $\pi_{P_G^\beta} : P_G^\beta \rightarrow U^\beta$ is the canonical projection of the bundle P_G^β ; the product between $g^{\alpha\beta}(\pi_{P_G^\beta}(p))$ and p is the group law of $G_{\pi_{P_G^\beta}(p)} \subset \mathcal{GL}(\mathcal{H}_S)$.

Let $\mathring{g}^{\alpha\beta} \in \mathcal{C}^\infty(U^\alpha \cap U^\beta, G)$ be the trivialization of $g^{\alpha\beta}$, i.e.

$$\mathring{g}^{\alpha\beta}(x) = (\zeta_x^\alpha)^{-1} \circ \xi_x^{\alpha\beta} \circ \zeta_x^\beta(1_G) = (\zeta_x^\alpha)^{-1}(g^{\alpha\beta}(x)\zeta_x^\beta(1_G)); \tag{C.16}$$

in other words, we have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\mathring{g}^{\alpha\beta}(x) \times} & G \\ \zeta_x^\beta \downarrow & & \downarrow \zeta_x^\alpha \\ \pi_{P_G^\beta}^{-1}(x) & \xrightarrow{\xi_x^{\alpha\beta}} & \pi_{P_G^\alpha}^{-1}(x) \\ & & n \end{array}$$

We define $\mathring{h}^{\alpha\beta\gamma} \in \mathcal{C}^\infty(U^\alpha \cap U^\beta \cap U^\gamma, J)$ by

$$\forall p \in \pi_{P_G^\alpha}^{-1}(x), \quad \xi_x^{\alpha\beta} \circ \xi_x^{\beta\gamma} \circ \xi_x^{\gamma\alpha}(p) = \mathring{h}^{\alpha\beta\gamma}(x)p; \tag{C.17}$$

in other words,

$$\mathring{h}^{\alpha\beta\gamma}(x) = (\zeta_x^\alpha)^{-1}(\mathring{h}^{\alpha\beta\gamma}(x)\zeta_x^\alpha(1_G)). \tag{C.18}$$

Let $\zeta_{x,\text{Lie}}^\alpha : \mathfrak{g} \rightarrow \mathfrak{g}_x$ be the map induced in the Lie algebras by ζ_x^α (the local trivialization of P_G^α). We extend ζ^α on the \mathfrak{g} -valued differential forms of U^α by $\zeta^\alpha : \Omega^*(U^\alpha, \mathfrak{g}) \rightarrow \Omega^*(U^\alpha, P_G^\alpha)$,

$$\zeta^\alpha(a_{\mu_1 \dots \mu_p}^i(x)X_i dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) = a_{\mu_1 \dots \mu_p}^i(x)\zeta_{x,\text{Lie}}^\alpha(X_i)dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \tag{C.19}$$

where $\{X_i\}_i$ is the set of the generators of \mathfrak{g} .

Property 23. ζ^α is a chain map and an algebra homomorphism: $\forall a, b \in \Omega^*(U^\alpha, \mathfrak{g})$,

$$d\zeta^\alpha(a) = \zeta^\alpha(da) \tag{C.20}$$

$$\zeta^\alpha(a \wedge b) = \zeta^\alpha(a) \wedge \zeta^\alpha(b), \tag{C.21}$$

the derivative on $\Omega^*(U^\alpha, P_G^\alpha) = \Omega^*(U^\alpha) \otimes_M \Gamma(U^\alpha, P_G^\alpha)$ being defined as $d \otimes 1_{P_G^\alpha}$.

Proof. This is a direct consequence of the definition of the extension of ζ^α . □

We are then able to define the trivializations of the different forms: $\mathring{A}^\alpha = (\zeta^\alpha)^{-1}(A^\alpha) \in \Omega^1(U^\alpha, \mathfrak{g})$, $\mathring{B}^\alpha = (\zeta^\alpha)^{-1}(B^\alpha) \in \Omega^2(U^\alpha, \mathfrak{j})$, $\mathring{F}^\alpha = (\zeta^\alpha)^{-1}(F^\alpha) \in \Omega^2(U^\alpha, \mathfrak{g})$, $\mathring{\eta}^{\alpha\beta} = (\zeta^\alpha)^{-1}(\eta^{\alpha\beta}) \in \Omega^1(U^\alpha \cap U^\beta, \mathfrak{j})$, $\mathring{\chi}^{\alpha\beta} = (\zeta^\alpha)^{-1}(\chi^{\alpha\beta}) \in \Omega^2(U^\alpha \cap U^\beta, \mathfrak{j})$ and $\mathring{H}^\alpha = (\zeta^\alpha)^{-1}(H^\alpha) \in \Omega^3(U^\alpha, \mathfrak{j})$.

The geometric structure \mathcal{P} is defined by the 1-transition $\mathring{g}^{\alpha\beta}$ and the 2-transition $\mathring{h}^{\alpha\beta\gamma}$. It is endowed with a 2-connection (see [29–31]) defined by the gauge potential \mathring{A}^α , the curving \mathring{B}^α and the potential transformation $\mathring{\eta}^{\alpha\beta}$. Its geometry is characterized by the fake curvature \mathring{F}^α , the curving transformation $\mathring{\chi}^{\alpha\beta}$ and the true curvature \mathring{H}^α . All these data define \mathcal{P} as a non-Abelian bundle gerbe [26–31] with the structure Lie crossed module (G, J, t, Ad) where $t : J \rightarrow G$ is the canonical injection (J is a subgroup of G) and $\text{Ad} : G \rightarrow \text{Aut}(J)$ is the adjoint representation of G on itself restricted to J (in the homomorphism domain).

C.2.2. Construction of \mathcal{P} as a categorical bundle. \mathcal{P} can be viewed as a principal categorical bundle (a 2-bundle, see [29–31]) on the left with the structure groupoid \mathcal{G} having $\text{Obj}(\mathcal{G}) = G$ as a set of objects and $\text{Morph}(\mathcal{G}) = J \rtimes G$ as a set of arrows, the semi-direct product (the horizontal composition of arrows) being defined by $(h, g)(h', g') = (h\text{Ad}(g)h', gg')$. The source map of \mathcal{G} is defined by $s(h, g) = g$ and the target map is defined by $t(h, g) = t(h)g$ (with $t(h)$ the canonical injection of h in G). The morphism composition (the vertical composition of arrows) is defined by $(h, g) \circ (h', t(h)g) = (hh', g)$.

The total category \mathcal{P} is then defined by $\text{Obj}(\mathcal{P}) = \bigsqcup_{\alpha} P_G^{\alpha}$ and $\text{Morph}(\mathcal{P}) = \bigsqcup_{\alpha} P_J^{\alpha} \times_M P_G^{\alpha}$ with $\forall (q, p) \in P_J^{\alpha} \times_M P_G^{\alpha}$ the source map $s(q, p) = p$ and the target map $t(q, p) = (\zeta_{\pi_{P_G^{\alpha}}(q)}^{\alpha})^{-1}(q)p$. The composition of arrows is $(q, p) \circ (q', \zeta_{\pi_{P_G^{\alpha}}(q)}^{\alpha})^{-1}(q)p = (qq', p)$. Let \mathcal{M} be the representation of M as a category, i.e. $\text{Obj}(\mathcal{M}) = M$ and $\text{Morph}(\mathcal{M}) = \{id_x : x \rightarrow x\}_{x \in M}$. The projection functor $\pi_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{M}$ is defined by $\pi_{\mathcal{P}}^{\text{Obj}} = \bigsqcup_{\alpha} \pi_{P_G^{\alpha}}$ and $\pi_{\mathcal{P}}^{\text{Morph}}(q, p) = id_{\pi_{P_G^{\alpha}}(p)}$. The local trivialization functor $\vartheta^{\alpha} : \mathcal{U}^{\alpha} \times \mathcal{G} \rightarrow \mathcal{P}_{|U^{\alpha}}$ is defined by $\vartheta^{\alpha, \text{Obj}} = \zeta^{\alpha}$ and $\vartheta^{\alpha, \text{Morph}} = \zeta_{|U^{\alpha} \times J}^{\alpha} \times \zeta^{\alpha}$ (\mathcal{U}^{α} is the representation of U^{α} as a category). Finally the left action functor $L : \mathcal{G} \rightarrow \text{Funct}(\mathcal{P}, \mathcal{P})$ of \mathcal{G} on \mathcal{P} ($\text{Funct}(\mathcal{P}, \mathcal{P})$ is the category of the endofunctors of \mathcal{P}) is defined as follows: $\forall g \in \text{Obj}(\mathcal{G}) = G$,

$$L(g) : \begin{array}{ccc} \text{Obj}(\mathcal{P}) & \rightarrow & \text{Obj}(\mathcal{P}) \\ P_G^{\alpha} \ni p & \mapsto & L_G^{\alpha}(g)p \end{array} \quad (\text{C.22})$$

and $\forall (h, g) \in \text{Morph}(\mathcal{G}) = J \rtimes G$,

$$L(h, g) : \begin{array}{ccc} \text{Morph}(\mathcal{P}) & \rightarrow & \text{Morph}(\mathcal{P}) \\ P_J^{\alpha} \times_M P_G^{\alpha} \ni (q, p) & \mapsto & (L_J^{\alpha}(h)\text{Ad}(L_G^{\alpha}(g))q, L_G^{\alpha}(g)p), \end{array} \quad (\text{C.23})$$

where L_J^{α} and L_G^{α} are the canonical left actions of J and G on P_J^{α} and P_G^{α} , respectively.

Finally, we can note that because G/J is a group since J is normal, \mathcal{P} is also a non-Abelian twisted bundle [36] associated with the extension of groups:

$$1 \rightarrow J \rightarrow G \rightarrow G/J \rightarrow 1.$$

Appendix D. About the adiabaticity of mixed states

Even if this is not strictly necessary, we consider in this section that the eigenoperator E is self-adjoint, in order to avoid unnecessary complications involving biorthogonal basis.

D.1. The adiabatic approximation

The following theorem shows that there exists an adiabatic approximation for the $*$ -eigenvectors.

Theorem 1 (Adiabatic approximation). *Let $M \ni x \mapsto E(x) = E_0(x) + \lambda(x)1_{\mathcal{H}_S} \in \mathcal{L}(\mathcal{H}_S)$ be a continuous self-adjoint eigenoperator decomposed following proposition 1, and $U^{\alpha} \ni x \mapsto \phi_{E_0, \lambda}^{\alpha}(x) \in \mathcal{H}_S \otimes \mathcal{H}_{\varepsilon}$ be an associated $*$ -eigenvector. Let $t \mapsto x(t) \in U^{\alpha}$ be an evolution. We assume the following adiabatic condition:*

$$\begin{aligned} & \forall \mu \in \text{Sp}(H - E_0 \otimes 1_{\mathcal{H}_{\varepsilon}}) \setminus \{\lambda\}, \\ & \forall \phi_{E_0, \mu} \in \ker(H - E_0 \otimes 1_{\mathcal{H}_{\varepsilon}} - \mu 1_{\mathcal{H}_S \otimes \mathcal{H}_{\varepsilon}}) \\ & \times \frac{\langle\langle \phi_{E_0, \mu}(x(t)) | \left(\frac{dH(x(t))}{dt} - \frac{dE_0(x(t))}{dt} \otimes 1_{\mathcal{H}_{\varepsilon}} \right) \phi_{E_0, \lambda}(x(t)) \rangle\rangle}{\lambda(x(t)) - \mu(x(t))} \simeq 0. \end{aligned} \quad (\text{D.1})$$

Then the wavefunction of the universe, $\psi(t)$ which is the solution of $i\hbar \frac{d\psi}{dt} = H(x(t))\psi(t)$ with $\psi(0) = \phi_{E_0, \lambda}^\alpha(x(0))$, rests within the $*$ -eigenspace associated with $E(x)$, i.e.

$$\forall t \quad \psi(t) \in \ker(H(x(t)) - E_0(x(t)) \otimes 1_{\mathcal{H}_\varepsilon} - \lambda(x(t))1_{\mathcal{H}_S \otimes \mathcal{H}_\varepsilon}). \quad (D.2)$$

Proof.

$$\begin{aligned} i\hbar \frac{d\psi}{dt} &= H\psi \\ \iff i\hbar \frac{d}{dt} (\mathbb{T} e^{-i\hbar^{-1} \int_0^t E_0(t') dt'} \tilde{\psi}) &= H \mathbb{T} e^{-i\hbar^{-1} \int_0^t E_0(t') dt'} \tilde{\psi}, \end{aligned} \quad (D.3)$$

with $\tilde{\psi} = \mathbb{T} e^{i\hbar^{-1} \int_0^t E_0(t') dt'} \psi$, $\mathbb{T}e$ being the time-ordered exponential (the Dyson series). We then have

$$\begin{aligned} E_0 \mathbb{T} e^{-i\hbar^{-1} \int_0^t E_0(t') dt'} \tilde{\psi} + i\hbar \mathbb{T} e^{-i\hbar^{-1} \int_0^t E_0(t') dt'} \frac{d\tilde{\psi}}{dt} &= H \mathbb{T} e^{-i\hbar^{-1} \int_0^t E_0(t') dt'} \tilde{\psi} \\ \Rightarrow i\hbar \frac{d\tilde{\psi}}{dt} &= \mathbb{T} e^{i\hbar^{-1} \int_0^t E_0(t') dt'} (H - E_0) \mathbb{T} e^{-i\hbar^{-1} \int_0^t E_0(t') dt'} \tilde{\psi} \\ \Rightarrow i\hbar \frac{d\tilde{\psi}}{dt} &= \widetilde{(H - E_0)} \tilde{\psi}. \end{aligned} \quad (D.4)$$

Let $\{\phi_{E_0, \lambda, a}\}_{\lambda \in \text{Sp}(H - E_0), a \in \{1, \dots, \text{deg}(\lambda)\}}$, be an orthonormal basis of the eigenspace of $H - E_0$ associated with λ . From the adiabatic condition, we have $\forall \lambda \neq \mu, \forall a, b$

$$\langle\langle \phi_{E_0, \mu, a} | \partial_t | \phi_{E_0, \lambda, b} \rangle\rangle = \frac{\langle\langle \phi_{E_0, \mu, a} | (\dot{H} - \dot{E}_0) \phi_{E_0, \lambda, b} \rangle\rangle}{\lambda - \mu} \simeq 0. \quad (D.5)$$

$\text{Sp}(\widetilde{H - E_0}) = \text{Sp}(H - E_0)$ with orthonormal eigenbasis $\{\tilde{\phi}_{E_0, \lambda, a} = \mathbb{T} e^{i\hbar^{-1} \int_0^t E_0(t') dt'} \phi_{E_0, \lambda, a}\}$. We then have $\forall \lambda \neq \mu, \forall a, b$

$$\begin{aligned} \langle\langle \tilde{\phi}_{E_0, \mu, a} | \partial_t | \tilde{\phi}_{E_0, \lambda, b} \rangle\rangle &= \frac{\langle\langle \tilde{\phi}_{E_0, \mu, a} | (\dot{\tilde{H}} - \dot{\tilde{E}}_0) \tilde{\phi}_{E_0, \lambda, b} \rangle\rangle}{\lambda - \mu} \\ &= \frac{\langle\langle \phi_{E_0, \mu, a} | (\dot{H} - \dot{E}_0) \phi_{E_0, \lambda, b} \rangle\rangle}{\lambda - \mu} \\ &\quad + i\hbar^{-1} \frac{\langle\langle \phi_{E_0, \mu, a} | \overbrace{[E_0, H]}^{=0} | \phi_{E_0, \lambda, a} \rangle\rangle}{\lambda - \mu} \\ &\simeq 0. \end{aligned} \quad (D.6)$$

We can then use the usual adiabatic approximation

$$\tilde{\psi}(t) = \sum_{a=1}^{\text{deg}(\lambda)} c_a(t) \tilde{\phi}_{E_0, \lambda, a}(t) \quad (D.7)$$

and then

$$\psi(t) = \sum_{a=1}^{\text{deg}(\lambda)} c_a(t) \phi_{E_0, \lambda, a}(t) \in \ker(H - E_0 - \lambda). \quad (D.8) \quad \square$$

The use of the adiabatic approximation is here heuristic. The study of rigorous adiabatic theorems is not the subject of this paper; nevertheless, this theorem shows that if the operator $H(x(t)) - E_0(x(t)) \otimes 1_{\mathcal{H}_\varepsilon}$ satisfies a usual adiabatic theorem, then the wavefunction of the universe stays in the $*$ -eigenspace.

Property 24. *The adiabatic condition (D.1) is equivalent to $R_\mu^\alpha(x(t))\dot{x}^\mu(t) \simeq 0$ ($\forall t$) where R^α is the remainder of the C^* -geometric phase generator.*

Proof. The set $\{\phi_{E_0,\lambda,a}\}_{\lambda \in \text{Sp}(H-E_0), a=1, \dots, \text{deg}(\lambda)}$, constitutes an orthonormal basis of $\mathcal{H}_S \otimes \mathcal{H}_E$; we then have

$$1 - P_{E_0,\lambda} = \sum_{\mu \neq \lambda} \sum_{a=1}^{\text{deg}(\mu)} |\phi_{E_0,\mu,a}\rangle \langle \phi_{E_0,\mu,a}| \quad (\text{D.9})$$

$$\begin{aligned} R_\mu \dot{x}^\mu \|\phi_{E_0,\lambda}\|^2 &= \text{tr}_{\mathcal{H}_E} ((1 - P_{E_0,\lambda}) |\partial_t \phi_{E_0,\lambda}\rangle \langle \phi_{E_0,\lambda}|) \\ &= \sum_{\mu \neq \lambda} \sum_a \langle \phi_{E_0,\mu,a} | \partial_t \phi_{E_0,\lambda} \rangle \\ &\quad \times \text{tr}_{\mathcal{H}_E} (|\phi_{E_0,\mu,a}\rangle \langle \phi_{E_0,\lambda}|) \\ &= \sum_{\mu \neq \lambda} \sum_a \frac{\langle \phi_{E_0,\mu,a} | (\dot{H} - \dot{E}_0 \otimes 1_{\mathcal{H}_E}) | \phi_{E_0,\lambda} \rangle}{\lambda - \mu} \\ &\quad \times \text{tr}_{\mathcal{H}_E} (|\phi_{E_0,\mu,a}\rangle \langle \phi_{E_0,\lambda}|) \\ &\simeq 0. \end{aligned} \quad (\text{D.10})$$

□

D.2. Some properties of the adiabatic transport formula

Property 25. *We have*

$$\forall t \quad \text{tr}_{\mathcal{H}_S} (g_{EA}(t) \rho_E^\alpha(x(t)) g_{EA}(t)^\dagger) = 1. \quad (\text{D.11})$$

Proof.

$$\frac{d}{dt} \text{tr}_{\mathcal{H}_S} (g_{EA} \rho_E g_{EA}^\dagger) = \text{tr}_{\mathcal{H}_S} (\dot{g}_{EA} \rho_E g_{EA}^\dagger) + \text{tr}_{\mathcal{H}_S} (g_{EA} \dot{\rho}_E g_{EA}^\dagger) + \text{tr}_{\mathcal{H}_S} (g_{EA} \rho_E \dot{g}_{EA}^\dagger) \quad (\text{D.12})$$

but

$$\dot{g}_{EA} = -g_{EA} (A_\mu + \eta_\mu) \dot{x}^\mu - i\hbar^{-1} E g_{EA} \quad (\text{D.13})$$

and

$$\dot{\rho}_E \simeq A_\mu \dot{x}^\mu \rho_E + \rho_E A_\mu^\dagger \dot{x}^\mu; \quad (\text{D.14})$$

then

$$\begin{aligned} \frac{d}{dt} \text{tr}_{\mathcal{H}_S} (g_{EA} \rho_E g_{EA}^\dagger) &= -\text{tr}_{\mathcal{H}_S} (g_{EA} (\eta_\mu \dot{x}^\mu \rho_E + \rho_E \eta_\mu^\dagger \dot{x}^\mu) g_{EA}^\dagger) \\ &\quad - i\hbar^{-1} \text{tr}_{\mathcal{H}_S} (E g_{EA} \rho_E g_{EA}^\dagger - g_{EA} \rho_E g_{EA}^\dagger E). \end{aligned} \quad (\text{D.15})$$

Moreover, we have

$$\eta_\mu \dot{x}^\mu \rho_E + \rho_E \eta_\mu^\dagger \dot{x}^\mu = 0 \quad (\text{D.16})$$

and

$$g_{EA} \phi_E \in \ker(H - E) \Rightarrow \mathcal{L}(g_{EA} \rho_E g_{EA}^\dagger) = E g_{EA} \rho_E g_{EA}^\dagger - g_{EA} \rho_E g_{EA}^\dagger E \quad (\text{D.17})$$

and since

$$\text{tr}_{\mathcal{H}_S} (\mathcal{L}(g_{EA} \rho_E g_{EA}^\dagger)) = \text{tr}_{\mathcal{H}_S} \text{tr}_{\mathcal{H}_E} ([H, g_{EA} |\phi_E\rangle \langle \phi_E| g_{EA}^\dagger]) = 0, \quad (\text{D.18})$$

we have

$$\mathrm{tr}_{\mathcal{H}_S}(E g_{EA} \rho_E g_{EA}^\dagger - g_{EA} \rho_E g_{EA}^\dagger E^\dagger) = 0. \quad (\text{D.19})$$

Finally, we have

$$\frac{d}{dt} \mathrm{tr}_{\mathcal{H}_S}(g_{EA} \rho_E g_{EA}^\dagger) = 0. \quad (\text{D.20})$$

□

This result is the parallel transport condition for the C^* -geometric phases.

Property 26. *The parallel transport condition is*

$$\frac{d}{dt} (\mathbb{P}_C e^{-\int_{x(0)}^{x(t)} (\mathcal{A}(x) + \eta(x))} \rho_E(x(t)) (\mathbb{P}_C e^{-\int_{x(0)}^{x(t)} (\mathcal{A}(x) + \eta(x))})^\dagger) = 0. \quad (\text{D.21})$$

Proof.

$$\begin{aligned} \frac{d}{dt} (g_{\mathcal{A}} \rho_E g_{\mathcal{A}}^\dagger) &= \dot{g}_{\mathcal{A}} \rho_E + g_{\mathcal{A}} \dot{\rho}_E g_{\mathcal{A}}^\dagger + g_{\mathcal{A}} \rho_E \dot{g}_{\mathcal{A}}^\dagger \\ &= -g_{\mathcal{A}} (\mathcal{A}_\mu + \eta_\mu) \dot{x}^\mu \rho_E g_{\mathcal{A}}^\dagger + g_{\mathcal{A}} (\mathcal{A}_\mu \dot{x}^\mu \rho_E + \rho_E \mathcal{A}_\mu^\dagger \dot{x}^\mu) g_{\mathcal{A}}^\dagger \\ &\quad - g_{\mathcal{A}} \rho_E (\mathcal{A}_\mu^\dagger + \eta_\mu^\dagger) \dot{x}^\mu g_{\mathcal{A}}^\dagger \\ &= -g_{\mathcal{A}} (\eta_\mu \dot{x}^\mu \rho_E + \rho_E \eta_\mu^\dagger \dot{x}^\mu) g_{\mathcal{A}} \\ &= 0. \end{aligned} \quad (\text{D.22})$$

□

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