Time stability characterization and spectral aliasing

Part II: a frequency-domain approach

F. Vernotte, G. Zalamansky and E. Lantz

Abstract. We showed in Part I of this paper [1] that the estimation of the white-phase noise level of a sampled signal may be achieved with variances even if the sampling frequency is far lower than the high cut-off frequency. In Part II, the effects of spectral aliasing for the different types of noise are reviewed. The responses of the Allan variance and the modified Allan variance for high-frequency noises are calculated taking into account the spectral aliasing. It is demonstrated that the effects of spectral aliasing for low-frequency noises may be neglected.

1. Introduction

The variances (Allan variance, modified Allan variance, ...) are powerful tools for spectral analysis. Variance measurements may easily be translated into noise-level measurements since the variance responses for all types of noises are well known. However, the sampling process applied over a noise may induce severe bias if it is not performed according to sampling theory, i.e. if the Nyquist frequency is lower than the high cut-off frequency of the analogue signal. In this case, spectral aliasing may completely alter the noise-level measurements.

We showed in a previous paper [1] these effects for a white-phase noise, using a time-domain approach. We presented a procedure which can recover both the white-phase noise level and the high cut-off frequency in some cases.

Part II of this paper deals with the effects of spectral aliasing for all types of noise (for power law exponents of the spectral density of phase from 0 to –5). In order to study these effects, a frequency-domain approach is used. The results for the white-phase noise are consistent with those obtained in [1] by using a time-domain approach. The responses of the Allan variance and the modified Allan variance are calculated versus the high cut-off frequency and the sampling frequency for the types of noise which are altered by spectral aliasing, i.e. the white-phase noise and the flicker-phase noise.

F. Vernotte: Observatoire de Besançon, 41 bis avenue de l'Observatoire, BP 1615, 25010 Besançon Cedex, France. e-mail: francois@obs-besancon.fr

G. Zalamansky: Université de Metz, Campus Bridoux, 57000 Metz, France. e-mail: zala@sciences.univ-metz.fr

E. Lantz: Laboratoire d'Optique P. M. Duffieux, UMR CNRS 6603, Faculté des Sciences et des Techniques, 25030 Besançon Cedex, France. e-mail: eric.lantz@univ-fcomte.fr

2. Theoretical study of spectral aliasing

2.1 General case: reminder of sampling theory

Let us consider the time error function $X(t)$ of an oscillator and $\tilde{X}(f)$ its Fourier transform. Let f_{h} be the high cut-off frequency, defined as the highest frequency for which $\tilde{X}(f)$ is non-zero (we discuss the value and the physical meaning of f_{h} for each type of noise). We showed in [1] that the high cut-off frequency f_{h} is the lower of the system high cut-off frequency and the noise process high cut-off frequency. This signal is sampled with a sampling frequency f_{sp}. Let $x(t)$ be an infinite sequence of samples of $X(t)$. This sequence $x(t)$ can be written as (see [2] or [3] for sampling theory)

$$x(t) = X(t) \times \sum_{i=-\infty}^{+\infty} \delta(f_{\mathrm{sp}}t - i), \tag{1}$$

where $\delta(t)$ is the Dirac distribution. The Fourier transform of this sequence is

$$\tilde{x}(f) = \tilde{X}(f) * \frac{1}{f_{\mathrm{sp}}} \sum_{i=-\infty}^{+\infty} \delta\left(\frac{f}{f_{\mathrm{sp}}} - i\right), \tag{2}$$

where * denotes a convolution product. Thus, for a frequency f_0 lower than the Nyquist frequency $f_{\mathrm{N}} = f_{\mathrm{sp}}/2$:

$$\tilde{x}(f_0) = \int_{-\infty}^{+\infty} \tilde{X}(f_0)\, \frac{1}{f_{\mathrm{sp}}} \times \sum_{i=-\infty}^{+\infty} \delta\left(\frac{f_0 - f}{f_{\mathrm{sp}}} - i\right) \mathrm{d}f = \sum_{i=-\infty}^{+\infty} \tilde{X}(f_0 + i f_{\mathrm{sp}}). \tag{3}$$

Then, whatever the frequency f_0, the Fourier transform $\tilde{x}(f)$ of $x(t)$ may be written as

$$\tilde{x}(f) = \tilde{X}(f) + \sum_{i\neq 0, i=-\infty}^{+\infty} \tilde{X}(f + i\, f_{\rm sp}) = \tilde{X}(f) + \Delta\tilde{X}(f). \quad (4)$$

The term $\Delta\tilde{X}(f)$ accounts for spectral aliasing. It can easily be verified that this term is null if $f_{\rm h} < f_{\rm sp}/2$ (sampling theorem). In the reverse case, ($f_{\rm sp} < 2f_{\rm h}$), $X(t)$ can no longer be recovered. However, as $X(t)$ is a random noise, some statistical properties of this noise can still be obtained. For the sake of simplicity, let us suppose that $f_{\rm h}/f_{\rm sp} = i_{\max} + 1/2$, where $i_{\max}$ is a finite integer. For $|f| < f_{\rm sp}/2$, (4) may be rewritten as (see Figure 1)

$$\tilde{x}(f) = \sum_{i=-i_{\max}}^{+i_{\max}} \tilde{X}(f + i\, f_{\rm sp}). \quad (5)$$

In this paper, we successively consider the case of white-phase noise, flicker-phase noise and low-frequency noises.

2.2 Case of white-phase noise

Let us consider the time error function $X(t)$ of an oscillator as a white Gaussian noise with a high cut-off frequency $f_{\rm h}$ (a white-phase noise without such a cut-off frequency could not exist because it would correspond to an infinite power). As the Fourier transform of a white Gaussian process is also a white Gaussian process, the Fourier transform of $X(t)$ is then a white Gaussian process of standard deviation $\sigma_{\rm SX}$. Therefore, the square of the modulus of its Fourier transform is chi-square distributed around the value $\sigma_{\rm SX^2}$. Thus, the spectral density $S_{\rm X}(f)$ of $X(t)$, which is the expectation of the square of the modulus of the Fourier transform of $X(t)$, may be modelled as

$$\begin{cases} S_X(f) = k_0\, f^0 = |\tilde{X}(f)|^2 = \sigma_{\rm SX^2} \\ \quad \text{if} \quad f \leq f_{\rm h} \\ S_X(f) = 0 \qquad \text{if} \quad f > f_{\rm h}, \end{cases} \quad (6)$$

where $\sigma_{\rm SX}$ is the standard deviation of $\tilde{X}(f)$ for $f \leq f_{\rm h}$.

Thus, from (5), the Fourier transform of $x(t)$ is the sum of $(2i_{\max} + 1)$ white Gaussian random numbers. The standard deviation of the sum of n independent gaussian white processes of standard deviation σ is $\sqrt{n}\,\sigma$. The standard deviation $\sigma_{\rm sx}$ of $\tilde{x}(f)$ is then

$$\sigma_{\rm sx} = \sqrt{2i_{\max} + 1}\,\sigma_{\rm SX}. \quad (7)$$

Let us define $s_x(f)$ as the expectation of the square of the modulus of the Fourier transform of $x(t)$. In the following, we refer to this quantity as the "apparent" spectral density. The differences between apparent and real spectral density are pointed out in [1]. Thus, $s_x(f)$ is then

$$s_x(f) = \sigma_{sx}^2 = (2i_{\max} + 1)\,\sigma_{\rm SX}^2 = (2i_{\max} + 1) S_{\rm X}(f) = 2\frac{f_{\rm h}}{f_{\rm sp}}\, S_{\rm X}(f). \quad (8)$$

Then, $x(t)$ is also a white Gaussian process, and its spectral density may be modelled as

$$s_x(f) = k_{0{\rm a}}\, f^0, \quad (9)$$

where

$$k_{0{\rm a}} = 2\,\frac{f_{\rm h}}{f_{\rm sp}} k_0. \quad (10)$$

Consequently, the sampling of a white-phase noise of level k_0 with a high cut-off frequency $f_{\rm h}$ and a sampling frequency $f_{\rm sp}$, yields a white-phase noise with a level $k_{0{\rm a}} = 2\,k_0\,f_{\rm h}/f_{\rm sp}$ and a high cut-off frequency equal to the Nyquist frequency $f_{\rm N} = f_{\rm sp}/2$ (of course, $s_x(f) = S_{\rm X}(f)$ if $f_{\rm h} = f_{\rm N}$ [2, 3]).

Therefore, after sampling a signal with a sampling frequency $f_{\rm sp}$, it is impossible to distinguish a white-phase noise of level k_0 and high cut-off frequency $f_{\rm h}$ from a white-phase noise of level Ak_0 and high cut-off frequency $f_{\rm h}/A$: the measured level is $2\,\frac{f_{\rm h}}{f_{\rm sp}}\,k_0$ in both cases (see Figure 2). Thus, neither the Allan

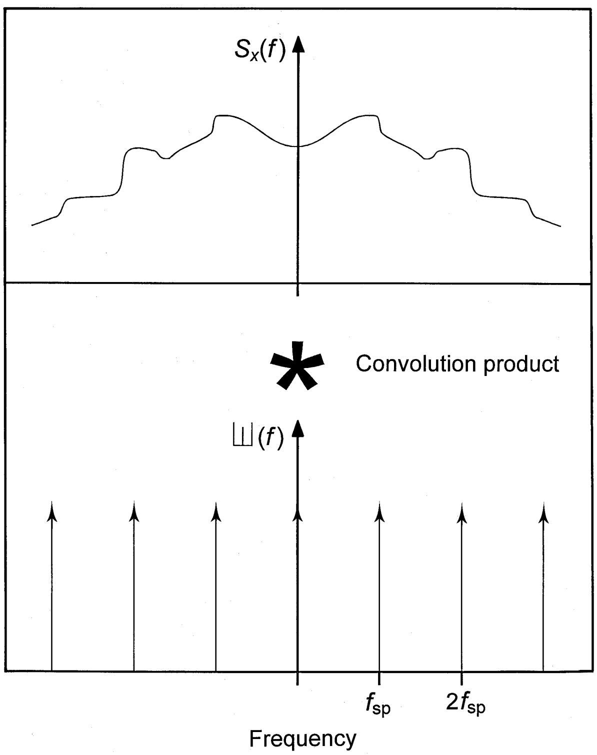

Figure 1. The sampling with a frequency $f_{\rm sp}$ induces spectral aliasing since each frequency sample f is the sum of the amplitudes of all frequencies $f + i\,f_{\rm sp}$ (i is an integer).

Table 1. Experimental measurements of the white-phase noise level ($k_0 = 0.5$, $f_\mathrm{h} = 100$ Hz, 8192 samples).

Sampling frequency f_sp/Hz	100	50	20	10	5
Measured level $k_{0\mathrm{a}}$	1.005 ± 0.8 %	1.952 ± 2.4 %	5.080 ± 1.5 %	9.981 ± 2.9 %	19.841 ± 1.3 %

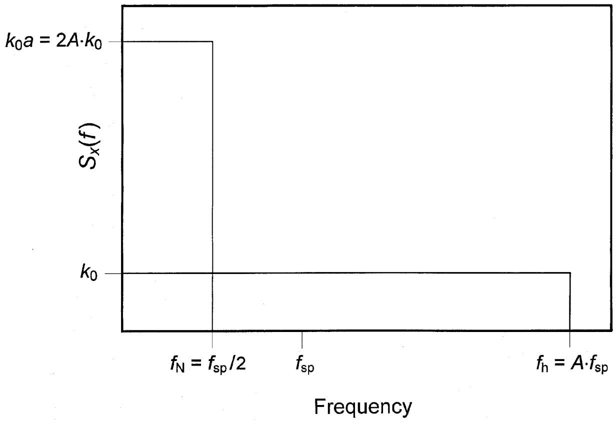

Figure 2. Sampling a white noise with a sampling frequency A times lower than the high cut-off frequency of this noise yields a measurement of the noise level $2A$ times greater than the real level. Furthermore, it is impossible to distinguish two noises of which the products $k_0 f_\mathrm{h}$ are the same.

variance, nor the modified Allan variance (nor any other variance, nor any other method) are able to measure the real white-phase level after sampling. The product $k_0 f_\mathrm{h}$ is the total noise power.

We experimentally confirmed this aliasing result by using a white-noise generator with a variable sampling frequency (see Table 1).

2.3 Case of flicker-phase noise

Let us consider the time error function $X(t)$ of an oscillator as a flicker-phase noise of level k_{-1} and high cut-off frequency f_h. The spectral density $S_\mathrm{X}(f)$ of $X(t)$ may be modelled as

$$S_\mathrm{X}(f) = k_{-1}\, f^{-1}. \tag{11}$$

The Fourier transform of $X(t)$ is a white Gaussian process $w(f)$ of standard deviation 1, multiplied by $\sqrt{k_{-1}\, f^{-1}}$:

$$\tilde{X}(f) = w(f)\,\sqrt{k_{-1}\, f^{-1}}. \tag{12}$$

After sampling, the Fourier transform of the sampled signal is (see Section 2.1, (5))

$$\tilde{x}(f) = \sum_{i=-i_\mathrm{max}}^{+i_\mathrm{max}} \tilde{X}(f + i\, f_\mathrm{sp}) = \sqrt{k_{-1}} \times \sum_i w(f + i\, f_\mathrm{sp})\,(f + i\, f_\mathrm{sp})^{-1/2} = \sqrt{k_{-1}} \sum_i w\,(f + i\, f_\mathrm{sp})\sigma_i, \tag{13}$$

where

$$\sigma_\mathrm{i} = (f + i\, f_\mathrm{sp})^{-1/2}. \tag{14}$$

The term $\sum_i w(f + i\, f_\mathrm{sp})\,\sigma_i$ is the sum of $(2i_\mathrm{max} + 1)$ Gaussian white processes of standard deviation σ_i. As the sum of two independent Gaussian white processes of standard deviation σ_1 and σ_2 is a Gaussian white process of standard deviation $\sqrt{\sigma_1^2 + \sigma_2^2}$, this sum may be rewritten as

$$\sum_i w(f + i\, f_\mathrm{sp})\,\sigma_i = w'(f) \times \sqrt{\sum_i \sigma_i^2} = w'(f)\,\sqrt{\sum_i (f + i\, f_\mathrm{sp})^{-1}}, \tag{15}$$

where $w'(f)$ is a white Gaussian process of standard deviation 1.
Then,

$$\tilde{x}(f) = \sqrt{k_{-1}}\, w'(f)\,\sqrt{\sum_i (f + i\, f_\mathrm{sp})^{-1}}. \tag{16}$$

Thus, the apparent spectral density $s_x(f)$ of the sampled signal $x(t)$ may be modelled as

$$s_x(f) = k_{-1} \sum_{i=-i_\mathrm{max}}^{+i_\mathrm{max}} |f + i\, f_\mathrm{sp}|^{-1} = k_{-1}\, f^{-1} + k_{-1} \sum_{i=-i_\mathrm{max}}^{-1} |f + i\, f_\mathrm{sp}|^{-1} + k_{-1} \sum_{i=+1}^{i_\mathrm{max}} |f + i\, f_\mathrm{sp}|^{-1}, \tag{17}$$

$$s_x(f) = S_\mathrm{X}(f) + \Delta S_x(f), \tag{18}$$

where

$$\Delta S_x(f) = k_{-1} \sum_{i=-i_\mathrm{max}}^{-1} |f + i\, f_\mathrm{sp}|^{-1} + k_{-1} \sum_{i=+1}^{i_\mathrm{max}} |f + i\, f_\mathrm{sp}|^{-1} = 2\, k_{-1} \sum_{i=+1}^{i_\mathrm{max}} (f + i\, f_\mathrm{sp})^{-1}, \tag{19}$$

$$\Delta S_x(f) = 2\, k_{-1}\, f_\mathrm{sp}^{-1} \sum_{i=+1}^{i_\mathrm{max}} \left(\frac{f}{f_\mathrm{sp}} + i\right)^{-1}, \tag{20}$$

with $0 \leq f/f_\mathrm{sp} \leq 1/2$. Moreover, it can be shown that, for $0 \leq \varepsilon \leq 1/2$,

$$\lim_{N \to \propto} \sum_{i=1}^{N} (\varepsilon + i)^{-1} = \ln(N). \tag{21}$$

Thus, if $f_{\rm h} \gg f_{\rm sp}$,

$$\Delta S_x(f) \approx 2\,k_{-1}\,f_{\rm sp}^{-1}\ln\left(\frac{f_{\rm h}}{f_{\rm sp}}\right) = k_0 = \text{constant}, \tag{22}$$

with

$$k_0 = \frac{2\,k_{-1}\ln\left(f_{\rm h}/f_{\rm sp}\right)}{f_{\rm sp}}. \tag{23}$$

Thus, after sampling a flicker-phase noise with a sampling frequency much lower than the high cut-off frequency of this noise, the flicker level is properly determined, but a white-phase noise also appears (see Figure 3). The higher the cut-off frequency, the greater the white-phase level. This level could even mask the flicker-phase level.

The approximation of relationship (21) is valid only for $f_{\rm h} \gg f_{\rm sp}$, but (23) is biased if $f_{\rm h}$ is close to $f_{\rm sp}$. For example, if the high cut-off frequency is equal to the Nyquist frequency, (23) yields a negative white level (which is nonsensical) whereas the signal is properly sampled and no white level appears. Similarly, if $f_{\rm sp} = f_{\rm h}$, (23) implies a null white level. However, a calculation with no approximation gives, in this case,

$$f_{\rm sp} = f_{\rm h} \qquad \text{and} \qquad 0 \le f \le \frac{f_{\rm sp}}{2} \Rightarrow$$

$$\frac{4k_{-1}}{3f_{\rm sp}} \le \Delta S_x(f) \le \frac{2k_{-1}}{f_{\rm sp}}. \tag{24}$$

In order to confirm our results, we simulated a flicker-phase noise with different high cut-off frequencies. Table 2 compares the measurements obtained with the multi-variance method and the levels calculated from (23). These results seem to confirm our theoretical considerations.

2.4 Case of low-frequency noises

Let us now consider the spectral density $S_{\rm X}(f)$ of the time error function of an oscillator affected by an f^α phase noise with $-5 \le \alpha \le -2$ (f^{-5} phase noise could affect millisecond pulsar timings):

$$S_{\rm X}(f) = k_\alpha\,f^\alpha. \tag{25}$$

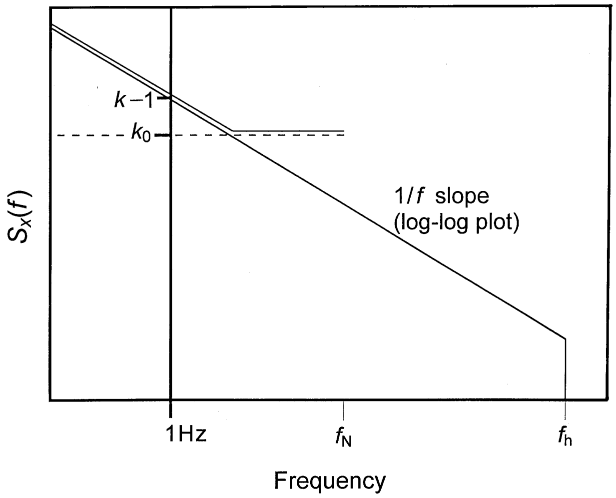

Figure 3. Sampling a flicker-phase noise with a high cut-off frequency higher than the Nyquist frequency yields a false white noise. The lower the Nyquist frequency, the higher the level of this noise.

The same calculations as those of the previous section show that the sampling, with a sampling frequency $f_{\rm sp}$, yields an apparent spectral density $s_x(f)$ as

$$s_x(f) = k_\alpha \sum_{i=-\infty}^{+\infty} |f + i\,f_{\rm sp}|^\alpha. \tag{26}$$

We do not consider here the high cut-off frequency because this sum converges for $\alpha \le -2$. We calculate this sum in the worst case, i.e. for an infinite high cut-off frequency,

$$s_x(f) = k_\alpha\,f^\alpha + 2\,k_\alpha \sum_{i=1}^{+\infty} (f + i\,f_{\rm sp})^\alpha, \tag{27}$$

$$s_x(f) = S_{\rm X}(f) + \Delta S_x(f), \tag{28}$$

where

$$\Delta S_x(f) = 2\,k_\alpha\,f_{\rm sp}^\alpha \sum_{i=1}^{+\infty}\left(\frac{f}{f_{\rm sp}} + i\right)^\alpha, \tag{29}$$

with $0 \le f/f_{\rm sp} \le 1/2$. Moreover, it can be shown that, for $0 \le \varepsilon \le 1/2$ and $\alpha \le -2$,

$$\lim_{N\to\infty} \sum_{i=1}^{N} (\varepsilon + i)^\alpha = 1. \tag{30}$$

Table 2. Experimental measurements of the false white-phase noise ($k_0 = 0$, $k_{-1} = 1$, $f_{\rm h} = 1$ Hz, 8192 samples). For high $f_{\rm h}/f_{\rm sp}$ values, the approximation in (21) becomes valid and the measured k_0 is close to the theoretical k_0.

$f_{\rm h}/f_{\rm sp}$	1	2	4	8	16	32	64	128	256
k_{-1} (measured)	0.909 ±7 %	1.03 ±4 %	0.985 ±2 %	0.991 ±4 %	0.991 ±3 %	0.981 ±5 %	0.946 ±6 %	1.03 ±7 %	0.904 ±9 %
k_0 (measured)	1.74 ±15 %	4.50 ±10 %	15.6 ±3 %	44.2 ±3 %	108 ±3 %	260 ±3 %	619 ±3 %	1446 ±3 %	3252 ±4 %
k_0 (theoretical)	0	2.77	11.1	33.3	88.7	222	532	1242	2839

Thus,

$$\Delta S_x(f) \approx 2\, k_\alpha\, f_{\mathrm{sp}}^{\alpha} = k_0 = \text{constant}. \tag{31}$$

As for the flicker-phase noise, the noise level is properly determined, but the sampling adds a false white-phase noise to the signal. We may calculate the ratio $R_{\mathrm{SX}}(f) = s_x(f)/S_{\mathrm{X}}(f)$ in order to estimate the influence of this effect. Obviously, this effect is greatest for the highest frequency, i.e. the Nyquist frequency $f_{\mathrm{N}} = f_{\mathrm{sp}}/2$.

$$R_{\mathrm{Sx}}(f_{\mathrm{N}}) = 1 + 2^{\alpha+1} \tag{32}$$

for $\alpha = -2 \quad R_{\mathrm{Sx}}(f_{\mathrm{N}}) = 1.5$
for $\alpha = -5 \quad R_{\mathrm{Sx}}(f_{\mathrm{N}}) = 1.0625$.

These results shows that the white-phase level is, at worst, of the same order as the f^α phase level for the higher frequencies. Thus, as we experimentally verified, this effect may be neglected for $\alpha \leq -2$. In the next section, we study the influence of spectral aliasing over variance responses for white-phase noise and flicker-phase noise only.

3. Responses of variances for high-frequency noises

The result of a variance with a transfer function $H(f)$ applied to a signal with a one-sided spectral density $S_y(f)$ may be calculated as [4]

$$\sigma_y^2(\tau) = \int_0^{+\infty} |H(f)|^2\, S_y(f)\mathrm{d}f = \int_0^{f_{\mathrm{sp}}} |H(f)|^2\, S'_y(f)\,\mathrm{d}f, \tag{33}$$

where $S'_y(f)$ is the apparent spectral density as defined in Section 2.2. In the following, we take into account the fact that the calculations of the variances are performed over the apparent spectral density.

3.1 The Allan variance

The square of the modulus of the transfer function of the Allan variance is [4]

$$|H_y(f)|^2 = 2\,\frac{\sin^4(\pi\tau\, f)}{(\pi\,\tau\, f)^2}. \tag{34}$$

The filter response of the Allan variance acts like an approximate constant-Q filter which analyses broadband power law spectra efficiently. The filter thus has a ratio of centre frequency to bandpass which is constant. One of the most attractive features of using the Allan variance is its ability to sort out various noises by the slopes on the Allan variance plot. However, the response from the sidelobes of (34) cause leakage in the case of white- and flicker-phase modulation (PM), making their slope indistinguishable and their level dependent on f_{h}. This paper is not concerned with leakage, but rather addresses the problem of undersampling.

3.1.1 Response for white-phase noise

In the case of a white-phase noise, the spectral density of the instantaneous normalized frequency deviation may be modelled as (without taking into account the high cut-off frequency)

$$S_y(f) = h_2\, f^{+2}. \tag{35}$$

The response of the Allan variance is then

$$\sigma_y^2(\tau) = \int_0^{+\infty} 2\,\frac{\sin^4(\pi\tau f)}{(\pi\tau f)^2} h_2\, f^{+2}\,\mathrm{d}f = \frac{2h_2}{(\pi\tau)^2}\int_0^{+\infty} \sin^4(\pi\tau f)\,\mathrm{d}f. \tag{36}$$

Since this integral does not converge for infinite frequencies, the high cut-off frequency must be taken into account. Assuming that $S_y(f)$ is null if $f > f_{\mathrm{h}}$, it becomes

$$\sigma_y^2(\tau) = \frac{2h_2}{(\pi\tau)^2}\int_0^{f_h} \sin^4(\pi\tau f)\,\mathrm{d}f = \frac{3\,h_2 f_h}{4\pi^2\tau^2}; \tag{37}$$

This result is the classical response of the Allan variance for a white-phase noise.

However, for a sampled signal, the high cut-off frequency must be considered as the Nyquist frequency and, translating (10) from phase-noise level k_0 into frequency-noise level h_2, the measured level is

$$h_{2\mathrm{a}} = 2\,\frac{f_{\mathrm{h}}}{f_{\mathrm{sp}}}\, h_2, \tag{38}$$

where f_{h} is the characteristic high cut-off frequency. Then, the response of the Allan variance for a white-phase noise may be rewritten as

$$\sigma_y^2(\tau) = \frac{3\,(2 f_{\mathrm{h}} h_2/f_{\mathrm{sp}})\, f_{\mathrm{N}}}{4\,\pi^2 - \tau^2}. \tag{39}$$

Since the sampling frequency is twice the Nyquist frequency:

$$\sigma_y^2(\tau) = \frac{3h_2 f_{\mathrm{h}}}{4\pi^2\tau^2}. \tag{40}$$

Thus, the expression of the Allan variance for a badly sampled white-phase noise is the same as the theoretical expression for an analogue computation.

3.1.2 Response for flicker-phase noise

In the case of a flicker-phase noise, the spectral density of the instantaneous normalized frequency deviation may be modelled as (without taking into account the high cut-off frequency)

$$S_y(f) = h_1\, f^{+1}. \tag{41}$$

The response of the Allan variance is then

$$\sigma_y^2(\tau) = \int_0^{+\infty} 2\,\frac{\sin^4(\pi\tau f)}{(\pi\tau f)^2} h_1\, f^{+1}\,\mathrm{d}f$$
$$= \frac{2\,h_1}{(\pi\tau)^2}\int_0^{+\infty}\frac{\sin^4(\pi\tau f)}{f}\,\mathrm{d}f. \qquad (42)$$

Since this integral does not converge for infinite frequencies, the high cut-off frequency must be taken into account. Assuming that $S_y(f)$ is null if $f > f_\mathrm{h}$, it becomes

$$\sigma_y^2(\tau) = \frac{2h_2}{(\pi\tau)^2}\int_0^{f_\mathrm{h}}\frac{\sin^4(\pi\tau f)}{f}\,\mathrm{d}f. \qquad (43)$$

This quantity may be approximated by

$$\sigma_y^2(\tau) = \frac{1.038 + 3\ln(2\,\pi\,f_\mathrm{h}\,\tau)}{4\,\pi^2\,\tau^2}\,h_1. \qquad (44)$$

This result is the classical response of the Allan variance for a flicker-phase noise.

For a sampled signal, however, the high cut-off frequency must be considered as the Nyquist frequency and a white noise level appears (see Section 2.3). Consequently, the result of the Allan variance is the same as the one obtained for a signal of which the high cut-off frequency is the Nyquist frequency and of which the spectral density may be modelled as

$$S'_y(f) = h_1\,f^{+1} + h_2\,f^{+2}, \qquad (45)$$

with, translating (23) from phase noise levels k_0 and k_{-1} to frequency noise level h_2 and h_1:

$$h_2 = \frac{2h_1\ln\left(\frac{f_\mathrm{h}}{f_\mathrm{sp}}\right)}{f_\mathrm{sp}}, \qquad (46)$$

where f_h is the characteristic high cut-off frequency.

Thus, the response of the Allan variance for a flicker-phase noise with a characteristic high cut-off frequency f_h and sampled with a sampling frequency f_sp is

$$\sigma_y^2(\tau) = \frac{1.038 + 3\ln(2\pi f_\mathrm{N}\tau)}{4\pi^2\tau^2}h_1 + \frac{3h_2 f_\mathrm{N}}{4\pi^2\tau^2}$$
$$= \frac{1.038 + 3\ln(2\pi f_\mathrm{N}\tau)}{4\pi^2\tau^2}h_1$$
$$+ \frac{3\left[2h_1\ln\left(f_\mathrm{h}/f_\mathrm{sp}\right)/f_\mathrm{sp}\right]f_\mathrm{N}}{4\pi^2\tau^2}. \qquad (47)$$

Since $f_\mathrm{N} = f_\mathrm{sp}/2$:

$$\sigma_y^2(\tau) = \frac{1.038 + 3\ln(\pi\,f_\mathrm{h}\,\tau)}{4\pi^2\tau^2}h_1. \qquad (48)$$

In this case also, the expression of the Allan variance for an undersampled flicker-phase noise is the same as the theoretical expression for an analogue computation.

Figure 4 shows the influence of the high cut-off frequency f_h on the Allan variance response, in the case of a flicker-phase noise.

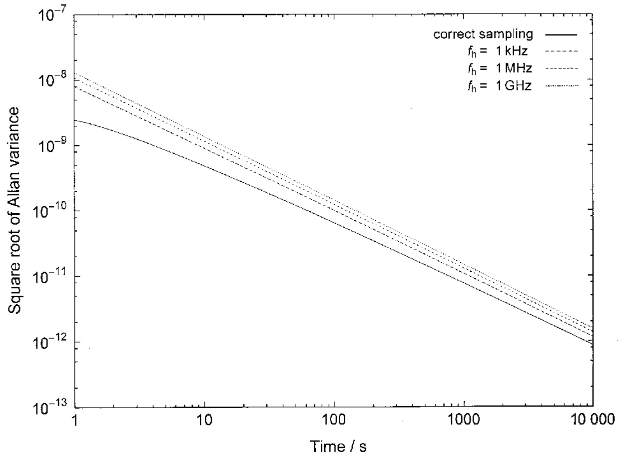

Figure 4. Influence of the high cut-off frequency f_h on the response of the Allan variance for a flicker-phase noise ($h_{+1} = 10^{-16}\,\mathrm{s}^2$, sampling frequency $f_\mathrm{sp} = 1\,\mathrm{Hz}$).

3.2 Modified Allan variance

The square of the modulus of the transfer function of the modified Allan variance is [5-7]

$$|H_\mathrm{M}(f)|^2 = 2\frac{\sin^6(\pi\tau f)}{(\pi\tau f)^2\,n^2\sin^2(\pi\tau_0 f)}, \qquad (49)$$

where τ_0 is the sampling period and n is defined as $n = \tau/\tau_0$.

This transfer function decreases more quickly versus frequency than the Allan variance. Thus, the modified Allan variance has significantly less leakage as evidenced in (49) above. The greater τ is (the greater n is), the faster this function decreases. Thus, if τ is greater than τ_0, only the amplitude of this function for the lowest frequencies is significant. Moreover, for the low frequencies, the following approximation may be used:

$$\pi\tau_0 f \ll \frac{\pi}{2} \quad\Rightarrow\quad \sin(\pi\tau_0 f) \approx \pi\tau_0 f. \qquad (50)$$

Then, if $\tau \gg \tau_0$, (49) may be rewritten as [8]:

$$|H_\mathrm{M}(f)|^2 = 2\,\frac{\sin^6(\pi\tau f)}{(\pi\tau f)^2 n^2(\pi\tau_0 f)^2}$$
$$= 2\,\frac{\sin^6(\pi\tau f)}{(\pi\tau f)^4}. \qquad (51)$$

The asymptotic behaviour (for large τ values) of the modified Allan variance for a signal with a spectral density $S_y(f)$, may be written as [7, 8]

$$\mathrm{mod}\,\sigma_y^2(\tau) = \int_0^{+\infty} 2\,\frac{\sin^6(\pi\tau f)}{(\pi\tau f)^4}S_y(f)\mathrm{d}f. \qquad (52)$$

3.2.1 Response for the white-phase noise

The asymptotic response of the modified Allan variance for a white-phase noise is

$$\mathrm{mod}\,\sigma_y^2(\tau) = \int_0^{+\infty} 2\,\frac{\sin^6(\pi\tau f)}{(\pi\tau f)^4} h_2 f^{+2} \mathrm{d}f = \frac{2h_2}{\pi^2\tau^2}\int_0^{+\infty} \frac{\sin^6(\pi\tau f)}{(\pi\tau f)^2}\mathrm{d}f. \quad (53)$$

This integral converges even for an infinite high cut-off frequency. The result is

$$\mathrm{mod}\,\sigma_y^2(\tau) = \frac{3h_2}{8\pi^2\tau^3}. \quad (54)$$

Thus, the asymptotic response (for large τ values) of the modified Allan variance for a white-phase noise does not depend on the high cut-off frequency. This result is true in the case of an analogue computation of the variance.

For a sampled signal, however, the real noise level must be replaced by the apparent noise level given in (38):

$$\mathrm{mod}\,\sigma_y^2(\tau) = \frac{3\,(2f_{\mathrm{h}}h_2/f_{\mathrm{sp}})}{8\pi^2\tau^3} = \frac{3h_2 f_{\mathrm{h}}}{4\pi^2 f_{\mathrm{sp}}\tau^3}. \quad (55)$$

The asymptotic response of the modified Allan variance for a badly sampled white-phase noise depends then on both high cut-off frequency and sampling frequency.

3.2.2 Response for the flicker-phase noise

For flicker-phase noise also, the modified Allan variance converges without taking into account the high cut-off frequency. Then, the asymptotic response of the modified Allan variance for a flicker-phase noise may be calculated without taking into account the high cut-off frequency:

$$\mathrm{mod}\,\sigma_y^2(\tau) = \frac{3\,[8\ln(2) - 3\ln(3)]}{8\pi^2\tau^2} h_1. \quad (56)$$

However, if this flicker-phase noise is badly sampled, a white-phase noise appears. Thus, the response of the modified Allan variance is

$$\mathrm{mod}\,\sigma_y^2(\tau) = \frac{3\,[8\ln(2) - 3\ln(3)]}{8\pi^2\tau^2} h_1 + \frac{3h_2}{8\pi^2\tau^3}. \quad (57)$$

From (46), this becomes

$$\mathrm{mod}\,\sigma_y^2(\tau) = \frac{3\,[8\ln(2) - 3\ln(3)]}{8\pi^2\tau^2} h_1 + \frac{3\,[2h_1 \ln(f_{\mathrm{h}}/f_{\mathrm{sp}})/f_{\mathrm{sp}}]}{8\pi^2\tau^3}. \quad (58)$$

Hence, the asymptotic response of the modified Allan variance for a flicker-phase noise with a characteristic high cut-off frequency f_{h} and sampled with a sampling frequency f_{sp} is

$$\mathrm{mod}\,\sigma_y^2(\tau) = \frac{3\left[8\ln(2) - 3\ln(3) + 2\ln(f_{\mathrm{h}}/f_{\mathrm{sp}})/(\tau f_{\mathrm{sp}})\right]}{8\pi^2\tau^2} h_1. \quad (59)$$

Figure 5 shows the influence of the high cut-off frequency f_{h} on the modified Allan variance response, in the case of a flicker-phase noise.

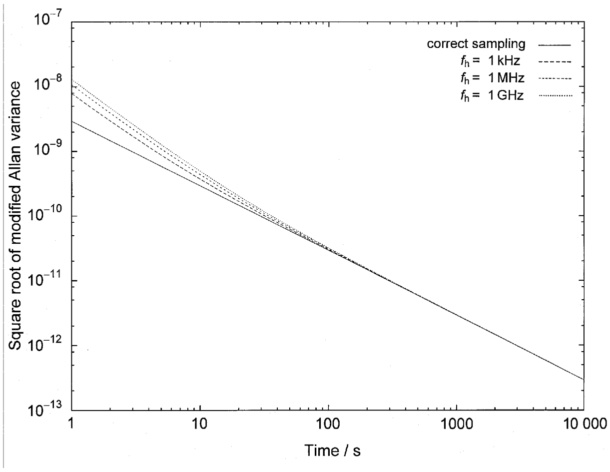

Figure 5. Influence of the high cut-off frequency f_{h} on the response of the modified Allan variance for a flicker-phase noise ($h_{+1} = 10^{-16}\,\mathrm{s}^2$, sampling frequency $f_{\mathrm{sp}} = 1\,\mathrm{Hz}$).

4. Conclusion

Since the effects of spectral aliasing are only noticeable for white-phase noise and flicker-phase noise, the responses of the variances were calculated versus the characteristic high cut-off frequency, which is the lower of the system cut-off frequency and the real cut-off frequency, taking into account:

- the increase of the white noise level in the case of white-phase noise;
- the white noise level which appears in the case of flicker-phase noise.

However, the effects of spectral aliasing, as a result of the sampling process, must be distinguished from the natural dependence of a variance on the cut-off frequency.

Table 3 gives the responses of the Allan variance and the modified Allan variance for white-phase noise and flicker-phase noise corrected for the effects of spectral aliasing. It can be seen that the response of the Allan variance is not modified. The responses of

Table 3. Transfer functions and responses of Allan variance and modified Allan variance for white-phase noise and flicker-phase noise without spectral aliasing (rows 3 and 4) and with spectral aliasing (rows 5 and 6).

$S_y(f)$	$\sigma_y^2(\tau)$	Mod $\sigma_y^2(\tau)$
$\vert H(f)\vert^2$	$2\dfrac{\sin^4(\pi\tau f)}{(\pi\tau f)^2}$	$2\dfrac{\sin^6(\pi\tau f)}{(\pi\tau f)^4}$
$h_{+2}\, f^{+2}$	$\dfrac{3h_{+2} f_\mathrm{h}}{4\pi^2\tau^2}$	$\dfrac{3h_{+2}}{8\pi^2\tau^3}$
$h_{+1}\, f^{+1}$	$\dfrac{1.038+3\ln(2\pi f_\mathrm{h}\tau)}{4\pi^2\tau^2}h_{+1}$	$\dfrac{3\left[8\ln(2)-3\ln(3)\right]}{8\pi^2\tau^2}h_{+1}$
$h_{+2}\, f^{+2}$	$\dfrac{3h_{+2} f_\mathrm{h}}{4\pi^2\tau^2}$	$\dfrac{3h_{+2} f_\mathrm{h}}{4\pi^2 f_\mathrm{sp}\tau^3}$
$h_{+1}\, f^{+1}$	$\dfrac{1.038+3\ln(\pi f_\mathrm{h}\tau)}{4\pi^2\tau^2}h_{+1}$	$3\left[\dfrac{8\ln(2)-3\ln(3)}{8\,\pi^2\tau^2} + \dfrac{\ln\left(\frac{f_\mathrm{h}}{f_\mathrm{sp}}\right)}{4\,\pi^2\tau^3 f_\mathrm{sp}}\right]h_{+1}$

other variances may be calculated for these noises also, using relationships (10) and (23).

These relationships may then be used in order to determine the real noise levels if the high cut-off frequency is known. However, if this frequency is unknown, only the product $k_0 f_\mathrm{h}$ may be measured in the case of a pure white-phase noise. For a flicker-phase noise mixed with a white phase-noise, it is impossible to know whether the white-phase noise level is a real one or a result of the spectral aliasing of the flicker-phase noise. However, the procedure given in [1] may solve this ambiguity.

On the other hand, it would be interesting to estimate the opposite effect: what happens in the case of low-frequency noise (flicker frequency noise and random-walk frequency noise) if the duration of the sampled sequence is far lower than the inverse of the low cut-off frequency of the signal. This will be the subject of a future paper.

References

1. Vernotte F., Zalamansky G., Lantz E., Time stability characterization and spectral aliasing, Part I: A time-domain approach, *Metrologia,* 1998, **35**, 723-730.
2. Roddier F., *Distributions et transformation de Fourier,* Paris, McGraw-Hill, 1984.
3. Oppenheim A. V., Schafer R. W., *Digital Signal Processing,* Engelwood Cliffs, Prentice Hall Inc., 1975.
4. Rutman J., Characterization of phase and frequency instabilities in precision frequency sources: fifteen years of progress, In *Proc. IEEE,* 1978, **66**, 1048-1075.
5. Allan D. W., Barnes J. A., A Modified “Allan variance” with increased oscillator characterization ability, In *Proc. 35th Ann. Frequency Control Symposium,* 1981, 470-474.
6. Lesage P., Ayi T., Characterization of frequency stability: analysis of the modified Allan variance and properties of its estimate, *IEEE Trans. Instrum. Meas.,* 1984, **IM-33**, 332-336.
7. Bernier L. G., Theoretical analysis of the modified Allan variance, In *Proc. 41st Ann. Frequency Control Symposium,* 1987, 116-121.
8. Vernotte F., *Stabilité temporelle des oscillateurs: nouvelles variances, leurs propriétés, leurs applications,* Ph.D. thesis, No. 199, 1991, Université de Franche-Comté, France.

Received on 12 January 1998 and in revised form on 3 June 1998.