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C^*-geometric phase for mixed states: entanglement, decoherence and the spin system

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Abstract

We study a kind of geometric phase for entangled quantum systems, and particularly a spin driven by a magnetic field and entangled with another spin. The new kind of geometric phase is based on an analogy between open quantum systems and dissipative quantum systems which uses a C^*-module structure. We show that the system presents, from the viewpoint of their geometric phases, two behaviours. The first one is identical to the behaviour of an isolated spin driven by a magnetic field, as the problem originally treated by Berry. The second one is specific to the decoherence process. The gauge structures induced by these geometric phases are then similar to a magnetic monopole gauge structure for the first case, and can be viewed as a kind of instanton gauge structure for the second case. We study the role of these geometric phases in the evolution of a mixed state, particularly by focusing on the evolution of the density matrix coherence. We investigate also the relation between the geometric phase of the mixed state of one of the entangled systems and the geometric phase of the bipartite system.

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1. Introduction

Since the pioneering works of Berry [1] and Simon [2] concerning the adiabatic dynamics of a closed two-level system, several types of geometric phases for quantum systems have been studied [3–8]. These geometric phase phenomena are related to pure states of a closed quantum system. For open quantum systems, four approaches have been proposed to extend the concept of the geometric phases to mixed states. (i) The purification of mixed states approach due to Uhlmann [9–11]. As the resulting purified states are Hilbert–Schmidt, a geometric phase can be associated with the evolution of vectors of the corresponding Hilbert space. (ii) An approach based on the decomposition of mixed states into convex combinations of pure states [12]. As for pure states, a geometric phase is the holonomy of the natural connection of a universal bundle over a complex projective space, the authors generalize this property considering the natural connection of the bundle induced by the convex combinations for the mixed state geometric phase. (iii) The Sarandy and Lidar approach [13, 14] which uses the Hilbert–Schmidt property of the reduced density matrices $\rho - I_n$ in finite space dimensions ($\dim \mathcal{H} = n$). The associated Lindblad equation can be considered as a Schrödinger equation in a n^2-dimensional Hilbert space. (iv) An approach due to Sjöqvist *et al* [15–17] which introduces a phase for evolving mixed states in interferometry and generalized to nonunitary evolution via the purification method. Recently, we have proposed a new approach [18] based on an analogy between the open and dissipative quantum systems realized by using a C^*-module structure [19]. The purpose of this paper is to study the role of this new geometric phase approach in entanglement and decoherence processes. We show that this new geometric phase, called the C^*-geometric phase, is related to the usual geometric phases of the entangled subsystems and with the geometric phase of the universe (the bipartite system). The relation is particularly significant for the viewpoint of statistical physics, since it is based on the statistical average of the C^*-geometric phase generator. Moreover, we show that the C^*-geometric phase is efficient to study the adiabatic control of a quantum system submitted to a decoherence process (an increase of the entanglement during the evolution). In order to enlighten the role of the C^*-geometric phases in a decoherence process, we study a very simple example which exhibits the main features associated with decoherence without other possible additional effects which could spoil the physical interpretation. This example is a spin driven by a magnetic field and entangled with the simple environment constituted by another spin. This system constitutes the more simple generalization to the open quantum systems of the example originally treated by Berry and Simon. The goal of this study is to compare the usual geometric phase with the new kind and their gauge structures. Indeed, it is known that the geometric phase of an isolated spin driven by a magnetic field $\vec{B}$ presents the gauge structure of a magnetic monopole in the space spanned by the three components of $\vec{B}$ [20]. We show in this paper that with respect to the considered mixed state, the gauge structure for a spin entangled with another one can be associated with a magnetic monopole or with a kind of instanton. The decoherence process is studied by considering the coherence of the system, i.e. the off-diagonal part of the density matrix. The decoherence process is characterized by a monotonic decrease of the coherence. In extreme cases, the coherence decreases to zero (for transitions from quantum to classical behaviours). In contrast, the other dynamical processes (as quantum transitions between eigenstates, modifications of the phases) which are also present without decoherence, induce oscillations in the coherence. We show that the C^*-geometric phase approach permits to geometrically describe separately the different phenomena. Indeed the decoherence process is associated with the instanton gauge structure whereas the other dynamical processes are associated with the magnetic monopole gauge structure.

The following section presents the concept and key results about the C^*-geometric phase theory exposed in [18], and explores the relation between the different geometric phases which can be defined for entangled systems. The following section is devoted to the derivation of the new kind of geometric phases for a spin model exhibiting a decoherence process. Finally, the last section explores the role of these geometric phases in adiabatic dynamics of a spin by studying the coherence of the system.

2. C^*-geometric phases and entanglement

2.1. The C^*-geometric phase theory

The C^*-geometric phases introduced in [18] are based on an analogy with the dissipative quantum systems. Let $H(x)$ be a parameter-dependent non-self-adjoint Hamiltonian in a

Hilbert space $\mathcal{H}$ and $\phi_\lambda(x)$ be one of its eigenvector (associated with a non-degenerate eigenvalue $\lambda(x)$). The adiabatic dynamics of the system involves the wavefunction $\psi(t) = e^{-\imath\hbar^{-1}\int_0^t \lambda(x(t'))dt'} e^{-\int_{\mathcal{C}} A_\lambda} \phi_\lambda(x(t))$ where $\mathcal{C}$ is the path drawn by $t \mapsto x(t)$ and the geometric phase is generated by $A_\lambda(x) = \frac{\langle\phi_\lambda|d\phi_\lambda\rangle}{\|\phi_\lambda\|^2}$ (we have supposed that $\psi(0) = \phi_\lambda(x(0))$). But if we are interested only in the dissipation process (the decreasing of the state norm), the main object is $\|\psi\|^2$ which plays the role of the state of dissipation of the quantum system. We have then[1] $\|\psi(t)\|^2 = e^{2\int_0^t \mathrm{Im}\lambda(x(t'))dt'} e^{-2\int_{\mathcal{C}} \mathrm{Re} A_\lambda} \|\phi_\lambda(x(t))\|^2$. The geometric contribution to the dissipation is then generated by $\mathrm{Re}\, A_\lambda = \frac{1}{2}\frac{d\|\phi_\lambda\|^2}{\|\phi_\lambda\|^2}$.

The paradigm of the C^*-geometric phase theory is that a dissipative quantum system can be viewed as a kind of open quantum system in which the dissipation is the only one effect of the coupling with the environment. $\|\psi\|^2$ for a dissipative system and the density matrix ρ for an open system play the same role. They characterize the influence of the environment on the system.

Let $\mathcal{H}_1$ be the Hilbert space of the system and $\mathcal{H}_2$ be the Hilbert space of the environment (in order to simplify the discussion, we assume that these spaces are finite dimensional). The dynamics of the universe (the bipartite system described by $\mathcal{H}_1 \otimes \mathcal{H}_2$) is governed by an Hamiltonian $H(x(t)) \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$:

$$H(x(t)) = H_1(x(t)) \otimes 1_{\mathcal{H}_2} + 1_{\mathcal{H}_1} \otimes H_2(x(t)) + H_I(x(t)), \quad (1)$$

where $H_i \in \mathcal{L}(\mathcal{H}_i)$ is the self-adjoint Hamiltonian of the free system i and $H_I \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is the system–environment interaction operator. $\mathcal{L}(\mathcal{H}_i)$ denotes the set of linear operators of $\mathcal{H}_i$. The vector x is a set of classical control parameters. Let $\psi(t)$ be a solution of the Schrödinger equation $\imath\hbar\dot{\psi} = H(x(t))\psi(t)$ for an evolution $t \mapsto x(t)$. The density matrix associated with the system is defined by the partial trace $\rho(t) = \mathrm{tr}_{\mathcal{H}_2} |\psi(t)\rangle\!\rangle\langle\!\langle\psi(t)|$. The scalar product in $\mathcal{H}_i$ will be denoted by $\langle .|.\rangle_i$ and the scalar product in $\mathcal{H}_1 \otimes \mathcal{H}_2$ will be denoted by $\langle\!\langle .|.\rangle\!\rangle$.

The analogy between dissipative quantum systems and entangled quantum systems is based on the following relationship between their models. The equivalent of the state space of the universe $\mathcal{H}_1 \otimes \mathcal{H}_2$ is $\mathbb{C} \otimes \mathcal{H} = \mathcal{H}$. We can then consider the open quantum system in the same manner as the dissipative quantum systems: we no longer consider $\mathcal{H}_1 \otimes \mathcal{H}_2$ as a vector space over the ring $\mathbb{C}$ but as a left C^*-module over the C^*-algebra $\mathcal{L}(\mathcal{H}_1)$. A module has the same axioms as a vector space but where an operator algebra takes the place of $\mathbb{C}$, see [19]. $\forall \psi, \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $\langle\psi|\phi\rangle_* = \mathrm{tr}_{\mathcal{H}_2} |\phi\rangle\!\rangle\langle\!\langle\psi|$ can then be considered as the inner product in the C^*-module (see [18]).

In this framework, we extend the notions of the eigenvector and eigenvalue to be consistent with the C^*-module structure: $\phi_E \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is said to be an $*$-eigenvector associated with the eigenoperator $E \in \mathcal{L}(\mathcal{H}_1)$ if

$$H\phi_E = E\phi_E \quad \text{and} \quad [E \otimes 1_{\mathcal{H}_2}, H]\phi_E = 0, \quad (2)$$

where the product $E\phi_E$ is defined as being the C^*-algebra action on its C^*-module ($E\phi_E \equiv E \otimes 1_{\mathcal{H}_2}\phi_E$). If $E = \lambda 1_{\mathcal{H}_1}$ with $\lambda \in \mathbb{R}$, then the $*$-eigenvector is a usual eigenvector of H. The eigenequation in the C^*-module is invariant under the action of two transformation groups: $G_x \subset \mathcal{GL}(\mathcal{H}_1)$ which is associated with operator-valued normalization changes and $K_x \subset \mathcal{U}(\mathcal{H}_1)$ which is associated with operator-valued phase changes [18] ($\mathcal{GL}(\mathcal{H}_1)$ is the set of invertible operators of $\mathcal{H}_1$ and $\mathcal{U}(\mathcal{H}_1)$ is the set of unitary operators of $\mathcal{H}_1$). These gauge changes have operator values since in the C^*-module structure the scalars of $\mathbb{C}$ are replaced by operators of the C^*-algebra $\mathcal{L}(\mathcal{H}_1)$. Throughout this paper, we assume that E is not degenerate in the sense defined in [18] which coincides with the usual sense if $E = \lambda 1_{\mathcal{H}_1}$.

[1] Since H is non-self-adjoint, their eigenvectors are not orthonormalized, but they constitute with the eigenvectors of $H^\dagger$ a biorthogonal basis. The normalization of ϕ_λ is then subject to a gauge choice (in contrast with the self-adjoint case where the gauge choice deals only with the phase of the eigenvector).

Table 1. Analogy between dissipative systems and entangled systems.

Dissipative quantum systems	Entangled quantum systems
Ring $\mathbb{C}$	C^*-algebra $\mathcal{L}(\mathcal{H}_1)$
Hilbert space $\mathbb{C}\otimes\mathcal{H}=\mathcal{H}$	C^*-module $\mathcal{H}_1\otimes\mathcal{H}_2$
$\|\psi\|^2 \in \mathbb{R}^{+*}$	$\|\psi\|_*^2 = \rho$
$H\phi_\lambda = \lambda\phi$ with $\lambda \in \mathbb{C}$	$H\phi_E = E\phi_E$ with $E \in \mathcal{L}(\mathcal{H}_1)$ and $[H,E]\phi_E = 0$
$A_\lambda = \frac{\langle\phi_\lambda\|\mathrm{d}\phi_\lambda\rangle}{\|\phi_\lambda\|^2}$	$\mathcal{A}_E\|\phi_E\|_*^2 = \langle\phi_E\|\mathrm{d}\phi_E\rangle_*$

The C^*-geometric phase generator is defined as being a solution of the following equation:

$$\mathcal{A}_E\|\phi_E\|_*^2 = \langle\phi_E|\mathrm{d}\phi_E\rangle_*$$
$$\iff \mathcal{A}_E\rho_E = \mathrm{tr}_{\mathcal{H}_2}|\mathrm{d}\phi_E\rangle\!\rangle\langle\!\langle\phi_E|, \qquad \mathcal{A}_E \in \mathcal{L}(\mathcal{H}_1), \tag{3}$$

where the 'mixed eigenstate' is $\rho_E = \mathrm{tr}_{\mathcal{H}_2}|\phi_E\rangle\!\rangle\langle\!\langle\phi_E|$. For a slow variation of the control parameters $t \mapsto x(t)$, by assuming an adiabatic assumption, we have, if $\psi(0) = \phi_E(x(0))$ [18],

$$\psi(t) = \mathbb{T}\,\mathrm{e}^{-\imath\hbar^{-1}\int_0^t E(x(t'))\mathrm{d}t'}\mathbb{P}\,\mathrm{e}^{-\int_{x(0)}^{x(t)}\mathcal{A}_E}\phi_E(x(t)) \tag{4}$$

and then

$$\rho(t) = g_E(t)g_{\mathcal{A}}(t)\rho_E(x(t))g_{\mathcal{A}}(t)^\dagger g_E(t)^\dagger, \tag{5}$$

with $g_E(t) = \mathbb{T}\mathrm{e}^{-\imath\hbar^{-1}\int_0^t E(x(t'))\mathrm{d}t'} \in \mathcal{L}(\mathcal{H}_1)$ being an $\mathcal{L}(\mathcal{H}_1)$-valued dynamical phase ($\mathbb{T}$e is the time-ordered exponential, i.e. the Dyson series [21]) and $g_{\mathcal{A}}(t) = \mathbb{P}\mathrm{e}^{-\int_{x(0)}^{x(t)}\mathcal{A}_E} \in \mathcal{L}(\mathcal{H}_1)$ an $\mathcal{L}(\mathcal{H}_1)$-valued geometric phase ($\mathbb{P}$e is the path-ordered exponential [20]). We call it the C^*-geometric phase since its values belong to the C^*-algebra $\mathcal{L}(\mathcal{H}_1)$. Table 1 summarizes the analogy between dissipative quantum systems and entangled quantum systems.

2.2. Relation between the different geometric phases of an entangled system

In [17], Tong *et al* study the relation between the different usual $U(1)$-valued geometric phases which can be defined in an entangled system and the geometric phase for mixed states of the Sjöqvist *et al* approach [15, 16]. In this section, we extend the results of [17] to an analysis involving the C^*-geometric phase approach.

By construction, we have

$$\mathrm{tr}_{\mathcal{H}_1}(\rho_E\mathcal{A}_E) = \mathrm{tr}_{\mathcal{H}_1}\mathrm{tr}_{\mathcal{H}_2}|\mathrm{d}\phi_E\rangle\!\rangle\langle\!\langle\phi_E| = \langle\!\langle\phi_E|\mathrm{d}\phi_E\rangle\!\rangle = A_{\mathcal{U}}, \tag{6}$$

where $A_{\mathcal{U}}$ is the generator of the $U(1)$-valued geometric phase associated with the state ϕ_E of the universe (the bipartite system). The generator of the geometric phase of the universe $A_{\mathcal{U}}$ is then the average of the C^*-geometric phase generator with respect to its mixed eigenstate ρ_E. By diagonalizing ρ_E,

$$\rho_E(x) = \sum_i p_i(x)|\chi_i(x)\rangle\langle\chi_i(x)| \quad p_i \in [0,1],\ \chi_i \in \mathcal{H}_1,\ \|\chi_i\|_1 = 1, \tag{7}$$

we find the $U(1)$-valued geometric phase of the universe which is

$$\int_{x(0)}^{x(t)} A_{\mathcal{U}} = \sum_i \int_{x(0)}^{x(t)} p_i(x)\langle\chi(x)|\mathcal{A}_E|\chi(x)\rangle_1 \tag{8}$$

This result exhibits the relation between the C^*-geometric phase of the mixed state with the $U(1)$-valued geometric phase of the universe. It is important to note that these two geometric phases are associated with two different adiabatic assumptions. The first one, a strong adiabatic assumption, states that $\psi(t)$ follows $\phi_E(x(t))$, i.e. $\forall t > 0$, $\psi(t) \in U(1) \cdot \phi_E(x(t))$ (in this expression '$\cdot$' denotes the group action). This strong assumption induces the $U(1)$-valued geometric phase: $\psi(t) = \mathrm{e}^{-\imath\hbar^{-1}\int_0^t \langle\langle \phi_E(x(t'))|H(x(t'))|\phi_E(x(t'))\rangle\rangle \mathrm{d}t'} \mathrm{e}^{-\int_{x(0)}^{x(t)} A_{\mathcal{U}}} \phi_E(x(t))$. The second one, a weak adiabatic assumption, states only that $\psi(t)$ remains into the eigenspace associated with E, i.e. $\forall t > 0$, $\psi(t) \in (G_{x(t)} \times K_{x(t)}) \cdot \phi_E(x(t))$. This weaker assumption induces the C^*-geometric phase (4). If $E \neq \lambda 1_{\mathcal{H}_1}$ the strong adiabatic assumption with ϕ_E cannot be satisfied since ϕ_E is not a usual eigenvector of H, but if $E = \lambda 1_{\mathcal{H}_1}$, then the two assumptions are equivalent (this point will be treated in the following section with an example of the spin system).

Equation (8) is a generalization of the results of [17]. Indeed to study only the effect of the entanglement on the geometric phase, as in [17] we consider the cases where $H_I = 0$. Hence $H = H_1 \otimes 1_{\mathcal{H}_1} + 1_{\mathcal{H}_2} \otimes H_2$ generates a bilocal unitary evolution, i.e. the entanglement degree is stationary; hence no decoherence process occurs. In this case, we can choose ϕ_E as being a Schmidt decomposition:

$$\phi_E(x) = \sum_i \sqrt{p_i} \zeta_{\mu_i}(x) \otimes \xi_{\nu_i}(x), \tag{9}$$

where $\sum_i p_i = 1$ with all coefficients p_i chosen independent of x, and $\zeta_{\mu_i} \in \mathcal{H}_1$ (respectively $\xi_{\nu_i} \in \mathcal{H}_2$) are normalized eigenvectors of H_1 (respectively H_2) associated with nondegenerate eigenvalues:

$$H_1 \zeta_{\mu_i} = \mu_i \zeta_{\mu_i} \qquad \mu_i \in \mathbb{R} \tag{10}$$

$$H_2 \xi_{\nu_i} = \nu_i \xi_{\nu_i} \qquad \nu_i \in \mathbb{R}. \tag{11}$$

The associated eigenoperator is

$$E(x) = \sum_i (\mu_i(x) + \nu_i(x)) |\zeta_{\mu_i}(x)\rangle\langle\zeta_{\mu_i}(x)| \tag{12}$$

satisfying $H\phi_E = E\phi_E$ and $[H, E \otimes 1_{\mathcal{H}_2}] = [H_1, E] = 0$. Also, we have

$$|\phi_E\rangle\rangle\langle\langle\phi_E| = \sum_{i,j} \sqrt{p_i p_j} |\zeta_{\mu_i}\rangle\langle\zeta_{\mu_j}| \otimes |\xi_{\nu_i}\rangle\langle\xi_{\nu_j}| \tag{13}$$

and consequently

$$\rho_E = \sum_i p_i |\zeta_{\mu_i}\rangle\langle\zeta_{\mu_i}|. \tag{14}$$

Equation (8) then takes the form

$$\int_{x(0)}^{x(t)} A_{\mathcal{U}} = \sum_i p_i \int_{x(0)}^{x(t)} \langle\zeta_{\mu_i}|\mathcal{A}_E|\zeta_{\mu_i}\rangle_1. \tag{15}$$

We note the difference with the Sjöqvist *et al* approach where the mixed state geometric phase $\arg\left(\sum_i p_i \mathrm{e}^{-\int_{x(0)}^{x(t')} a_i}\right)$ is 'submitted to interferences between individual geometric phases' and which is not directly related to the geometric phase of the universe [17]. The a_i are $U(1)$-valued geometric phase generators which are not $\langle\zeta_{\mu_i}|\mathcal{A}_E|\zeta_{\mu_i}\rangle_1$, see [17].

The equation defining the C^*-geometric phase $\mathcal{A}_E \rho_E = \langle\phi_E|\mathrm{d}\phi_E\rangle_*$ has for the solution (if $\forall i$, $p_i \neq 0$),

$$\mathcal{A}_E = \sum_i |\mathrm{d}\zeta_{\mu_i}\rangle\langle\zeta_{\mu_i}| + \sum_{i,j} \sqrt{\frac{p_i}{p_j}} \langle\xi_{\nu_i}|\mathrm{d}\xi_{\nu_j}\rangle_2 |\zeta_{\mu_i}\rangle\langle\zeta_{\nu_j}|. \tag{16}$$

We see that $\mathcal{A}_E$ depends not only on the $U(1)$-geometric phases associated with the individual states but also on nonadiabatic transition factors $\langle \xi_{\nu_i} | \mathrm{d}\xi_{\nu_j} \rangle_2$ $(j \neq i)$. This is in accordance with the fact that the adiabatic assumption associated with the C^*-geometric phase is weaker than a strict adiabatic assumption in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Finally, equation (15) becomes

$$\int_{x(0)}^{x(t)} A_{\mathcal{U}} = \sum_i p_i \left(\int_{x(0)}^{x(t)} \langle \zeta_{\mu_i} | \mathrm{d}\zeta_{\mu_i} \rangle_1 + \int_{x(0)}^{x(t)} \langle \xi_{\nu_i} | \mathrm{d}\xi_{\nu_i} \rangle_2 \right). \tag{17}$$

We recover in the context of the C^*-geometric phase the result of Tong *et al* [17] concerning the relation between the $U(1)$-valued geometric phase of the universe and the $U(1)$-valued geometric phases of the two subsystems[2].

3. C^*-geometric phases of a spin system

In order to exhibit the role of the C^*-geometric phase in decoherence processes, we treat a simple system in which the signature of the decoherence is easily recognizable. This section is devoted to the description of this system in the framework of the C^*-geometric phases.

3.1. The model

We consider a spin-1/2 $\vec{S}_1$ driven by a magnetic field $\vec{B}$ and entangled with another spin-1/2 $\vec{S}_2$. This system is governed by the Hamiltonian

$$H(\underline{B}) = \vec{B} \cdot \vec{S}_1 + \frac{\alpha}{\hbar} \vec{S}_1 \cdot \vec{S}_2, \tag{18}$$

where $\alpha > 0$ is the strength of the coupling between the system ($\vec{S}_1$) and its environment ($\vec{S}_2$). H depends on parameters described by the quadrivector $\underline{B} = (B^0, B^1, B^2, B^3)$ with $\vec{B} = (B^1, B^2, B^3)$ and $B^0 = \sqrt{\|\vec{B}\|^2 + \alpha^2}$. (The choice of B^0 in place of α to represent the decoherence parameter will become clear in the following.) The interest of the system is that it is the more simple superposition of a driven system ($\vec{B} \cdot \vec{S}_1$) with an interaction inducing a decoherence process ($\frac{\alpha}{\hbar} \vec{S}_1 \cdot \vec{S}_2$).

We are interested in time variations of $\underline{B}$ sufficiently slow in order to ensure that the dynamics of the universe (the system plus its environment) rests adiabatic. Following the adiabatic approximation [21], the wavefunction of the universe can be described by using the $\underline{B}$-dependent eigenvectors of $H(\underline{B})$. The spectrum of $H(\underline{B})$ is constituted by $\lambda_1 = \frac{\hbar}{4}(\alpha - 2B)$, $\lambda_2 = \frac{\hbar}{4}(\alpha + 2B)$, $\lambda_3 = \frac{\hbar}{4}(-\alpha - 2B^0)$ and $\lambda_4 = \frac{\hbar}{4}(-\alpha + 2B^0)$ ($B = \|\vec{B}\| = \sqrt{(B^1)^2 + (B^2)^2 + (B^3)^2}$). The eigenvectors associated with λ_1 and λ_2 are in the basis $(|\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle)$:

$$\phi_{\lambda_1} = \frac{1}{2B(B + B^3)} \begin{pmatrix} (B^2 + \imath B^1)^2 \\ (B^1 - \imath B^2)(B + B^3) \\ (B^1 - \imath B^2)(B + B^3) \\ -(B + B^3)^2 \end{pmatrix} \tag{19}$$

and

$$\phi_{\lambda_3} = \frac{1}{2\sqrt{B^0(B^0 + \alpha)}} \begin{pmatrix} B^2 + \imath B^1 \\ -\imath(B^0 + B^3 + \alpha) \\ \imath(B^0 - B^3 + \alpha) \\ B^2 - \imath B^1 \end{pmatrix}. \tag{20}$$

[2] In [16], the cyclic case is different from the noncyclic case for the nonadiabatic geometric phases; this difference does not occur for the adiabatic geometric phases which involve eigenvectors.

The eigenvector associated with λ_2 (respectively λ_4) is obtained from ϕ_{λ_1} (respectively ϕ_{λ_3}) by changing B into $-B$ (respectively B^0 into $-B^0$). We can note that the two first eigenvectors ϕ_{λ_1} and ϕ_{λ_2} are independent from the coupling strength α.

These eigenvectors belong to the Hilbert space of the universe $\mathcal{H}_1 \otimes \mathcal{H}_2$ (where $\mathcal{H}_i \simeq \mathbb{C}^2$ are the spin state spaces). They are associated with eigen-density matrices of the system $\rho_{\lambda_j} = \mathrm{tr}_{\mathcal{H}_2} |\phi_{\lambda_j}\rangle\langle\phi_{\lambda_j}|$ which are for ϕ_{λ_1} and ϕ_{λ_3}

$$\rho_{\lambda_1} = \frac{1}{2B}\begin{pmatrix} B - B^3 & -B^1 + \imath B^2 \\ -B^1 - \imath B^2 & B + B^3 \end{pmatrix} \tag{21}$$

and

$$\rho_{\lambda_3} = \frac{1}{2B^0}\begin{pmatrix} B^0 - B^3 & -B^1 + \imath B^2 \\ -B^1 - \imath B^2 & B^0 + B^3 \end{pmatrix}. \tag{22}$$

3.2. The adiabatic magnetic monopole

The generators of the C^*-geometric phases $\mathcal{A}_{\lambda_j}$ can then be defined[3] in a complete analogy with the dissipative quantum systems as being the solutions of

$$\begin{aligned} \mathcal{A}_{\lambda_j} \|\phi_{\lambda_j}\|_*^2 &= \langle \phi_{\lambda_j} | \mathrm{d}\phi_{\lambda_j} \rangle_* \\ \mathcal{A}_{\lambda_j} \rho_{\lambda_j} &= \mathrm{tr}_{\mathcal{H}_2} \left(|\mathrm{d}\phi_{\lambda_j}\rangle\langle\phi_{\lambda_j}| \right). \end{aligned} \tag{23}$$

First we consider the case associated with λ_1 (the case associated with λ_2 is equivalent). Since $\det(\rho_{\lambda_1}) = 0$, ρ_{λ_1} is not invertible and equation (23) has several solutions (which are related by a gauge transformation specific to the open quantum systems, see [18]). This degeneracy could be associated with the fact that ϕ_{λ_1} is an eigenvector independent from α. Let $(|\hat{\uparrow}\rangle, |\hat{\downarrow}\rangle)$ be the diagonalization basis of ρ_{λ_1} $\left(\hat{\rho}_{\lambda_1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)$. We denote by M the change of matrix basis between $(|\uparrow\rangle, |\downarrow\rangle)$ and $(|\hat{\uparrow}\rangle, |\hat{\downarrow}\rangle)$. We can note that the eigenvector of the universe is in the basis $(|\hat{\uparrow}\rangle, |\hat{\downarrow}\rangle)$,

$$\begin{aligned} \hat{\phi}_{\lambda_1} &= M^{-1} \otimes 1_{\mathcal{H}_2} \phi_{\lambda_1} \\ &= \frac{1}{\sqrt{2B(B + B^3)}} \begin{pmatrix} -B^1 - \imath B^2 \\ B + B^3 \\ 0 \\ 0 \end{pmatrix} \end{aligned} \tag{24}$$

which is identical to the eigenvector of $\vec{B} \cdot \vec{S}_1$ in the basis $(|\uparrow\rangle, |\downarrow\rangle)$. In the new basis we then have

$$\begin{aligned} \hat{\mathcal{A}}_{\lambda_1} \hat{\rho}_{\lambda_1} &= \langle \widehat{\phi_{\lambda_1} | \mathrm{d}\phi_{\lambda_1}} \rangle_* \\ &= \langle \hat{\phi}_{\lambda_1} | \mathrm{d}\hat{\phi}_{\lambda_1} \rangle_* + M^{-1} \mathrm{d}M \hat{\rho}_{\lambda_1} \end{aligned} \tag{25}$$

with $\langle \hat{\phi}_{\lambda_1} | \mathrm{d}\hat{\phi}_{\lambda_1} \rangle_* = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where

$$A = -\frac{\imath}{2} \frac{B^2 \mathrm{d}B^1 - B^1 \mathrm{d}B^2}{B(B + B^3)} \tag{26}$$

is the generator of the geometric phase originally studied by Berry (the Berry potential). It corresponds to the magnetic potential of a magnetic monopole living in the space $\mathbb{R}^3$

[3] *Remark:* in [18] we define eigenoperators in place of the eigenvalues. Such operators, $E \in \mathcal{L}(\mathcal{H}_1)$, obey $H\phi_E = E\phi_E$ and $[H, E] = 0$. This is consistent with the paradigm where the ring $\mathbb{C}$ is replaced by the C^*-algebra $\mathcal{L}(\mathcal{H}_1)$. But in this work, the condition $[H, E] = 0$ induces that E must commute with each generator of the Lie algebra $\mathfrak{su}(2)$ being in its irreducible representation $j = \frac{1}{2}$ (i.e. $\vec{S}_1$). Now by the Schur lemma this induces that E is a multiple of the identity. The notion of the eigenoperator is then reduced to the usual notion of the eigenvalue for the system studied in this paper.

spanned by $\vec{B}$ (see [20]). The magnetic monopole has a magnetic charge $1/2$ and is located at $B^1 = B^2 = B^3 = 0$ (the Dirac string being $(0, 0, B^3)$ for $B^3 < 0$). It follows that

$$\hat{\mathcal{A}}_{\lambda_1} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + M^{-1}dM + \eta, \tag{27}$$

where $\eta = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$ is an arbitrary 1-form ($*$ denotes arbitrary numbers) which constitutes a gauge transformation specific to the open quantum systems (it is associated with K_x, see [18]). Finally, $\mathcal{A}_{\lambda_1} = M\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}M^{-1} + dMM^{-1} + M\eta M^{-1}$ is modulo the gauge transformations (M and η) the same geometric phase generator as that for an isolated spin.

3.3. The adiabatic instanton

We consider now the case associated with λ_3 (the case associated with λ_4 is equivalent). In that case $\det(\rho_{\lambda_3}) = \left(\frac{\alpha}{2B^0}\right)^2 \neq 0$. It is easy to see that the solution of equation (23) is[4] $\mathcal{A}_{\lambda_3} = \frac{1}{2}\mathrm{d}\rho_{\lambda_3}\rho_{\lambda_3}^{-1}$. The computation shows that

$$\mathcal{A}_{\lambda_3} = -\frac{\sigma^0\,\mathrm{d}B^0}{2B^0} + \frac{C_{\mu\nu\rho}B^\mu\sigma^\nu\,\mathrm{d}B^\rho}{2\eta_{\mu\nu}B^\mu B^\nu}, \tag{28}$$

where we have adopted the Einstein convention on the repetition of two greek indices (they belong to $\{0, 1, 2, 3\}$). $\{\sigma^i\}_{i=1,2,3}$ are the Pauli matrices and σ^0 is the identity 2×2 matrix. $\eta_{\mu\nu}$ is the Minkwoski metric ($\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$) and $C_{\mu\nu\rho}$ is a kind of 't Hooft symbol

$$\begin{cases} C_{\mu\nu 0} = \eta_{\mu\nu} \\ C_{ijk} = \imath\epsilon_{ijk} \\ C_{0ij} = -C_{i0j} = \delta_{ij}, \end{cases} \tag{29}$$

where the latin indices belong to $\{1, 2, 3\}$, δ_{ij} is the Kronecker symbol and ϵ_{ijk} is the Lévi–Cività symbol.

$\mathcal{A}_{\lambda_3}$ can be viewed as a potential of a kind of instanton (see [20, 22]) living in the Minkowski space-time $\mathbb{R}^{3+1}$ spanned by $\underline{B}$. The strength of the coupling between the two spins $\alpha = \sqrt{\eta_{\mu\nu}B^\mu B^\nu}$ is then the 'proper time' between the instanton and the 'event' associated with $\underline{B}$ (which is by construction timelike). The reason for which we call the gauge structure an 'adiabatic instanton' is the following. Usually, an instanton is a vacuum solution of an Euclidean action. In our development, the 'space-time' is Minkowskian and not Euclidean, and no classical action appears. But we focus on the following fundamental property of the instanton fields $A_{\mathrm{inst}}(r)$: for r (the Euclidean distance) in the neighbourhood of $+\infty$, $A_{\mathrm{inst}}(r)$ is asymptotically pure gauge. Now, for α in the neighbourhood of $+\infty$, the couplings between the two spins $\frac{\alpha}{\hbar}\vec{S}_1\cdot\vec{S}_2$ outclass the effect of the magnetic field. It follows that $\mathcal{A}_{\lambda_3}$ is asymptotically pure gauge. Indeed[5] $\phi_{\lambda_3} \sim g(\underline{B})(0, -1, 1, 0)$ where $g(\underline{B}) = \mathrm{e}^{\imath\varphi(\underline{B})}$ is an arbitrary $\underline{B}$-dependent phase choice, and then $\mathcal{A}_{\lambda_3} \sim \mathrm{d}g^{-1}g$ is pure gauge (if we keep the phase choice of equation (22), we have $\mathcal{A}_{\lambda_3} \sim 0$ which is just a particular gauge choice). This property is very similar to the case of a usual instanton (except that α is a Minkowskian distance).

[4] We have $\mathrm{d}\rho_\lambda = \mathrm{tr}_{\mathcal{H}_2}(|\mathrm{d}\phi_\lambda\rangle\langle\phi_\lambda| + |\phi_\lambda\rangle\langle\mathrm{d}\phi_\lambda|) = \mathcal{A}_\lambda\rho_\lambda + \rho_\lambda\mathcal{A}_\lambda^\dagger$; moreover, $\mathrm{d}\rho_\lambda = \frac{1}{2}(\mathrm{d}\rho_\lambda\rho_\lambda^{-1}\rho_\lambda + \rho_\lambda(\mathrm{d}\rho_\lambda\rho_\lambda^{-1})^\dagger)$. By identification, we can set $\mathcal{A}_\lambda = \frac{1}{2}\mathrm{d}\rho_\lambda\rho_\lambda^{-1}$, which is also the C^*-geometric phase generator modulo a gauge transformation specific to the open quantum systems (see [18]).

[5] *Remark*: ϕ_{λ_4} diverges when $\alpha \to +\infty$ but with another phase convention the divergence is when $\alpha \to -\infty$. This is similar to the problem of the Dirac strings in the magnetic monopole gauge structure [20].

4. The coherence of the spin system

In this section, we study the role of the C^*-geometric phases for an adiabatic dynamics induced by a slow variation $t \mapsto \underline{B}(t)$ in the spin system. We suppose that the wavefunction of the universe is initially $\psi(0) = a\phi_{\lambda_1}(\underline{B}(0)) + b\phi_{\lambda_3}(\underline{B}(0))$ with $a, b \in \mathbb{C}$ ($|a|^2 + |b|^2 = 1$). It is natural to consider that the system is in a generic state and not in an eigenstate since it seems difficult in practice to prepare an open quantum system in an eigen-density matrix which is a very particular mixed state (for the sake of simplicity we have ignored the couple $(\phi_{\lambda_2}, \phi_{\lambda_4})$ without loss of generality since it is physically redundant with $(\phi_{\lambda_1}, \phi_{\lambda_3})$). We assume the adiabatic assumption stating that the wavefunction of the universe is

$$\psi(t) = a\,\mathrm{e}^{-\imath\hbar^{-1}\int_0^t \lambda_1 \mathrm{d}t'}\mathbb{P}\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \mathcal{A}_{\lambda_1}}\phi_{\lambda_1}(\underline{B}(t)) + b\,\mathrm{e}^{-\imath\hbar^{-1}\int_0^t \lambda_3 \mathrm{d}t'}\mathbb{P}\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \mathcal{A}_{\lambda_3}}\phi_{\lambda_3}(\underline{B}(t)), \tag{30}$$

where $\mathbb{P}\mathrm{e}$ is the path ordering exponential [20] along the path drawn by $t \mapsto \underline{B}(t)$ (it corresponds to a time-ordered exponential, i.e. a Dyson series). The action of $\mathbb{P}\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \mathcal{A}_{\lambda_1}}$ on ϕ_{λ_1} is the action defined by the C^*-module (i.e. $\mathbb{P}\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \mathcal{A}_{\lambda_1}}\phi_{\lambda_1} \equiv \mathbb{P}\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \mathcal{A}_{\lambda_1}} \otimes 1_{\mathcal{H}_2}\phi_{\lambda_1}$). The time-dependent density matrix is then

$$\begin{aligned}\rho(t) = |a|^2\rho_{\lambda_1} + |b|^2\rho_{\lambda_3} + a\bar{b}\,\mathrm{e}^{-\imath\hbar^{-1}\int_0^t(\lambda_3-\lambda_1)\mathrm{d}t'}\mathbb{P}\,\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \mathcal{A}_{\lambda_1}}\tau_{\lambda_1\lambda_3}\left(\mathbb{P}\,\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \mathcal{A}_{\lambda_3}}\right)^\dagger \\ + \bar{a}b\,\mathrm{e}^{\imath\hbar^{-1}\int_0^t(\lambda_3-\lambda_1)\mathrm{d}t'}\mathbb{P}\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \mathcal{A}_{\lambda_3}}\tau_{\lambda_3\lambda_1}\left(\mathbb{P}\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \mathcal{A}_{\lambda_1}}\right)^\dagger,\end{aligned} \tag{31}$$

where $\tau_{\lambda_i\lambda_j} = \mathrm{tr}_{\mathcal{H}_2}|\phi_{\lambda_i}\rangle\langle\phi_{\lambda_j}|$. Since $[\rho_{\lambda_1}, \rho_{\lambda_3}] = 0$, ρ_{λ_1} and ρ_{λ_3} are simultaneously diagonalizable. $\hat{\rho}_{\lambda_3} = \frac{1}{2B^0}\begin{pmatrix} B^0+B & 0 \\ 0 & B^0-B\end{pmatrix}$. We then work in the basis $(|\hat{\uparrow}\rangle, |\hat{\downarrow}\rangle)$ which is the natural basis if we want to consider that the spin $\vec{S}_1$ has the behaviour of an isolated spin for the state ϕ_{λ_1} (which is independent from α the coupling strength with the environment). In this basis, we set $\hat{\tau}_{\lambda_1\lambda_3} = \begin{pmatrix} d & c \\ \bar{c} & -d\end{pmatrix}$, $d \in \mathbb{R}$ and $c \in \mathbb{C}$. We have

$$\frac{1}{2}\mathrm{d}\hat{\rho}_{\lambda_3}\hat{\rho}_{\lambda_3}^{-1} = \begin{pmatrix} \hat{\mathcal{A}}_{\lambda_3\uparrow} & 0 \\ 0 & \hat{\mathcal{A}}_{\lambda_3\downarrow}\end{pmatrix}, \tag{32}$$

where

$$\hat{\mathcal{A}}_{\lambda_3\downarrow} = \frac{1}{2}\frac{B\mathrm{d}B^0 - B^0\mathrm{d}B}{B^0(B^0-B)}. \tag{33}$$

$\hat{\mathcal{A}}_{\lambda_3\uparrow}$ is obtained by changing $B \to -B$. We can note that this form is exact: $\hat{\mathcal{A}}_{\lambda_3\uparrow} = \mathrm{d}\ln\sqrt{\frac{B^0-B}{B^0}}$. Nevertheless, $\sqrt{\frac{B^0-B}{B^0}}$ is not a phase but a non-zero real number (another terminology, as geometric factor for example, could be more appropriate); in consequence, an elimination of the instanton geometric phase $\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)}\hat{\mathcal{A}}_{\lambda_3\downarrow}}$ needs a 'gauge change' consisting in modifying the normalization of ϕ_{λ_3}. Such a transformation does not preserve the trace of ρ_{λ_3} and is then not a valid operation. We can compare with the dissipative quantum systems, where the geometric contribution to the dissipation is an exact generator $\mathrm{Re}\,A = d\ln\|\phi_\lambda\|$ which has a physical meaning. In contrast with the usual geometric phases, $\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)}\hat{\mathcal{A}}_{\lambda_3\downarrow}}$ is physically relevant for the open paths $t \mapsto \underline{B}(t)$ and is 1 for the closed paths.

We then consider the density matrix in the basis $(|\hat{\uparrow}\rangle, |\hat{\downarrow}\rangle)$, $\hat{\rho}(t) = M(\underline{B}(t))^{-1}\rho(t)M(\underline{B}(t))$. The coherence of the system, $\hat{c}_{\uparrow\downarrow} = \langle\hat{\uparrow}|\hat{\rho}(t)|\hat{\downarrow}\rangle$, is then

$$\hat{c}_{\uparrow\downarrow}(t) = |abc(\underline{B}(t))|\mathrm{e}^{-\int_{\underline{B}(0)}^{\underline{B}(t)}\hat{\mathcal{A}}_{\lambda_3\downarrow}}\cos\left(-\hbar^{-1}\int_0^t(\lambda_3-\lambda_1)\,\mathrm{d}t' - \imath\int_{\underline{B}(0)}^{\underline{B}(t)} A + \varphi_{abc}(t)\right), \tag{34}$$

where $\varphi_{abc}(t) = \arg a - \arg b + \arg c(\underline{B}(t))$. The evolution of the coherence is then driven by two geometric phases: $\mathrm{e}^{-\int A}$, the geometric phase associated with the magnetic monopole gauge structure which is responsible with the dynamical phase for the oscillations of the coherence, and $e^{-\int \hat{\mathcal{A}}_{\lambda_3\downarrow}}$, the geometric phase associated with the instanton gauge structure which is responsible for the amplitude variations of the coherence. We have

$$e^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \hat{\mathcal{A}}_{\lambda_3\downarrow}} = \sqrt{\frac{B^0(t)}{B^0(t)-B(t)} \frac{B^0(0)-B(0)}{B^0(0)}}. \tag{35}$$

If $\frac{\alpha(t)}{B(t)}$ becomes small, then $e^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \hat{\mathcal{A}}_{\lambda_3\downarrow}} \sim \frac{B(t)}{\alpha(t)} \gg 1$ and the coherence is large which is a characteristic of a quantum superposition of $|\hat{\uparrow}\rangle$ and $|\hat{\downarrow}\rangle$. If $\alpha(t)$ grows sufficiently, then $e^{-\int_{\underline{B}(0)}^{\underline{B}(t)} \hat{\mathcal{A}}_{\lambda_3\downarrow}} \simeq 1$ and the coherence is minimal which is characteristic of a mixed state closer to a 'classical' mixture of $|\hat{\uparrow}\rangle$ and $|\hat{\downarrow}\rangle$. This high decoherence effect is driven on the wavefunction by the geometric phase associated with the instanton structure. Moreover, we can remark that the $U(1)$-valued geometric phase generators of the universe are $A_{\mathcal{U},1} = \langle\hat{\uparrow}|\hat{\mathcal{A}}_{\lambda_1}|\hat{\uparrow}\rangle = A$ and $A_{\mathcal{U},3} = \left(\frac{1}{2}+\frac{B}{B^0}\right)\hat{\mathcal{A}}_{\lambda_3\uparrow} + \left(\frac{1}{2}-\frac{B}{B^0}\right)\hat{\mathcal{A}}_{\lambda_3\downarrow} = 0$. The magnetic monopole gauge structure is present at the level of the universe (a closed system), in accordance with the fact that it is associated with the usual dynamical processes. In contrast, the instanton gauge structure is only present at the level of the subsystem $\vec{S}_1$, in accordance with the fact that it characterizes the decoherence (a relation between $\vec{S}_1$ and $\vec{S}_2$).

5. Conclusion

From the viewpoint of the geometric phases, a spin driven by a magnetic field and entangled with another spin presents two distinct behaviours. For two eigenstates, it is similar to an isolated spin driven by a magnetic field (the original system studied by Berry) and its gauge structure is then similar to a magnetic monopole in the space $\mathbb{R}^3$. It is associated only with oscillations of the coherence. For the two other eigenstates, the geometric phase is specific to the decoherence process induced by the entanglement, and its gauge structure is similar to the kind of instanton in a Minkowski space-time $\mathbb{R}^{3+1}$. It is associated with the high decoherence effects, as the transition from quantum superpositions to classical mixtures. Finally, we can make an interesting remark. The previous approaches of geometric phases for open quantum systems, as for example the Sarandy–Lidar approach [13, 14], provide geometric phases which are associated with a combination of the usual dynamical effects and the decoherence process. They then seem not very efficient to interpret and analyse the decoherence effects. In contrast, the C^*-geometric phases theory seems to be able to provide geometric phases specific to the usual dynamical effects (as A being the magnetic monopole gauge potential), and geometric phases specific to the decoherence process (as $\mathcal{A}_{\lambda_3}$ being the instanton gauge potential). It would be interesting to study the C^*-geometric phases for more complicated systems.

References

[1] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[2] Simon B 1983 *Phys. Rev. Lett.* **51** 2167
[3] Wilczek A and Zee A 1984 *Phys. Rev. Lett.* **52** 2111
[4] Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 1593
[5] Samuel J and Bhandari R 1988 *Phys. Rev. Lett.* **60** 2339
[6] Moore D J 1990 *J. Phys. A: Math. Gen.* **23** L665
[7] Mostafazadeh A 1999 *Phys. Lett.* A **264** 11
[8] Viennot D 2009 *J. Math. Phys.* **50** 052101
[9] Uhlmann A 1986 *Rep. Math. Phys.* **24** 229
[10] Uhlmann A 1993 *Rep. Math. Phys.* **33** 253

[11] Uhlmann A 1995 *Rep. Math. Phys.* **36** 461
[12] Chaturvedi S, Ercolessi E, Marmo G, Morandini G, Mukunda N and Simon R 2004 *Eur. Phys. J.* C **35** 413
[13] Sarandy M S and Lidar D A 2006 *Phys. Rev.* A **73** 062101
[14] Sarandy M S, Duzzioni E I and Moussa M H Y 2007 *Phys. Rev.* A **76** 052112
[15] Sjöqvist E, Pati A K, Ekert A, Anandan J S, Ericsson M, Oi D K L and Vedral V 2000 *Phys. Rev. Lett.* **85** 2845
[16] Tong D M, Sjöqvist E, Kwek L C and Oh C H 2004 *Phys. Rev. Lett.* **93** 080405
[17] Tong D M, Sjöqvist E, Kwek L C, Oh C H and Ericsson M 2003 *Phys. Rev.* A **68** 022106
[18] Viennot D and Lages J 2011 *J. Phys. A: Math. Theor.* **44** 365301
[19] Landsman N P 1998 arXiv:math-ph/9807030
[20] Nakahara M 1990 *Geometry, Topology and Physics* (Bristol: Institute of Physics Publishing)
[21] Messiah A 1959 *Quantum Mechanics* (Paris: Dunod)
[22] Itzykson C and Zuber J B 1980 *Quantum Field Theory* (New York: Dover)