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Generalized composition law from 2×2 matrices

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Many results that are difficult can be found more easily by using a generalization in the complex plane of Einstein's addition law of parallel velocities. Such a generalization is a natural way to add quantities that are limited to bounded values. We show how this generalization directly provides phase factors such as the Wigner angle in special relativity and how this generalization is related in the simplest case to the composition of 2×2 S-matrices.
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I. INTRODUCTION

In special relativity the composition law of parallel velocities appears to be the natural addition law for quantities whose values are limited to the closed interval $[-1,1]$, where we have set the speed of light $c=1$. It is natural to generalize the composition law of parallel velocities to the complex plane as

$$A = A_2 \oplus A_1 = \frac{A_1 + A_2}{1 + \overline{A_2} A_1}, \tag{1}$$

where A_1 and A_2 are complex quantities and where the denominator appears as a normalization term (if not otherwise stated, $\bar{A}$ denotes the complex conjugation operation). The physical meaning of this composition law is similar to that of the composition law of parallel velocities in special relativity. Equation (1) shows that no matter what real values we give to $A_1=v_1$ and $A_2=v_2$, subject only to $v_1<c$ and $v_2<c$, the value of the resulting velocity $A=w$ cannot exceed the speed of light $c=1$. In the same way, no matter the values of the complex quantities A_1 and A_2 (subject only to $|A_1|<1$ and $|A_2|<1$), the modulus of the resulting quantity A cannot exceed unity.

Because it avoids infinities, such a generalization of Einstein's composition law of velocities appears to be a natural addition law in a closed interval. As expected, it reduces to the usual addition of arithmetics when the quantities are small. As shown in Refs. 1–3, the use of this composition law quickly leads to important theoretical results and provides useful algorithms for computer calculations. The use of Eq. (1) also leads to results that converge more rapidly than by using transfer matrices.[4]

II. SOME SIMPLE EXAMPLES

We consider three examples[1–3] where the use of Eq. (1) is useful. The examples are the composition of two nonparallel velocities in special relativity, the reflection coefficient of a Fabry–Pérot in optics, and the characteristics of a polarizer resulting from the association of two successive nonperfect polarizers.

The composition law for two parallel velocities v_1 and v_2 in special relativity is $(c=1)$

$$w = v_2 \oplus v_1 = \frac{v_1 + v_2}{1 + v_1 v_2}. \tag{2}$$

The calculation of the resulting velocity of two parallel velocities is straightforward. However, it is not when the two velocities are not parallel for which calculations may be tedious. They become simple when we consider Eq. (1), which is the generalization in the complex plane of Eq. (2). As explained in Ref. 1, we replace each velocity $\vec{v_i}$ by the complex number,

$$V_i = \tanh\frac{a_i}{2} e^{i\alpha_i}, \tag{3}$$

where the rapidity a_i is related to the modulus of $\vec{v_i}$ by $\tanh a_i = v_i$ and where phase α_i gives the orientation of $\vec{v_i}$ with respect to an arbitrary axis of the reference frame of the observer in the plane of $\vec{v_1}$ and $\vec{v_2}$. The modulus and phase α of the velocity $\vec{w}$ resulting from the relativistic composition of $\vec{v_1}$ and $\vec{v_2}$ are directly obtained[1] by using Eq. (1),

$$W = \tanh\frac{a}{2} e^{i\alpha} = V_2 \oplus V_1 = \frac{\tanh\frac{a_1}{2} e^{i\alpha_1} + \tanh\frac{a_2}{2} e^{i\alpha_2}}{1 + \tanh\frac{a_2}{2} e^{-i\alpha_2} \tanh\frac{a_1}{2} e^{i\alpha_1}}. \tag{4}$$

The modulus and the phase of Eq. (4) give respectively the magnitude of the resulting velocity $\vec{w}$ (because $w=\tanh a$) and specify the direction α of $\vec{w}$ in the plane $(\vec{v_1},\vec{v_2})$.

In optics the overall reflection coefficient of a Fabry–Pérot interferometer can be obtained by taking into account all virtual paths of light inside the interferometer.[2] The total probability amplitude for light to be reflected by the system can also be directly obtained (for any number of interfaces) by using Eq. (1). Here

$$R_i = r_i e^{i\phi_i} \tag{5}$$

is the complex reflection coefficient of an incident wave on interface i, where r_i is the Fresnel coefficient of that interface and ϕ_i is the phase shift corresponding to the propagation of light through the same homogeneous layer between two successive interfaces. For two interfaces the reflection coefficient of the whole system can be obtained directly by using law (1),

$$R = re^{i\phi} = R_2 \oplus R_1 = \frac{r_1 e^{i\phi_1} + r_2 e^{i\phi_2}}{1 + r_2 e^{-i\phi_2} r_1 e^{i\phi_1}}. \tag{6}$$

Again, the modulus and the phase of Eq. (6) give the overall reflection coefficient and phase of the reflected wave.

Similarly, we can consider the composition of two nonperfect polarizers P_1 and P_2.[3] The polarizer P resulting from the combination of polarizers P_1 and P_2 (in that order) can also be obtained from Eq. (1). As explained in Ref. 3, each polarizer is characterized by

$$P_i = \tanh\frac{\gamma_i}{2} e^{i\alpha_i}, \tag{7}$$

where γ_i is the quality of the polarizer and α_i gives the orientation of the polarizer axis with respect to an arbitrary reference axis. Typically, $\gamma_i = \Gamma_i z$, where Γ_i is the differential absorption rate of the polarizer and z is the distance traveled by the light wave inside the polarizer. The case of a perfect polarizer[5] corresponds to $\gamma_i \to +\infty$. The polarizer's orientations and the reference axis are coplanar. The characteristics of the resulting polarizer P are given by[3]

$$P = \tanh\frac{\gamma}{2} e^{i\alpha} = P_2 \oplus P_1 = \frac{\tanh\frac{\gamma_1}{2} e^{i\alpha_1} + \tanh\frac{\gamma_2}{2} e^{i\alpha_2}}{1 + \tanh\frac{\gamma_1}{2} e^{-i\alpha_1} \tanh\frac{\gamma_2}{2} e^{i\alpha_2}}. \tag{8}$$

By using the composition law (8) we easily extract the γ factor and its direction α.

The use of the composition law (1) is general and can be applied to any number of coplanar velocities in special relativity, to any number of interfaces for the case of multilayers, and to any number of successive polarizers. In such cases we have to iterate Eq. (1) as relation[2,3,6]

$$A = A_n \oplus (A_{n-1} \oplus \cdots (A_2 \oplus A_1)). \tag{9}$$

The successive iteration of Eq. (1) yields the desired result. Equation (9) leads to algorithms that are useful for many problems. It is easy to compute $A_2 \oplus A_1$ and then to compose the result with A_3 and so on.

As explained in Refs. 7 and 2, the expression for A in Eq. (9) can be written down directly by using a complex generalization of the elementary symmetric functions of the variables $A_1, A_2, \ldots, A_n$, which are extensively used in the theory of polynomials.[8,9]

Our aim in this paper is to show how Eq. (1) is related to 2×2 matrices and how it provides a simple way to calculate the four elements of *scattering matrices* (S-matrices). We also show how the use of Eq. (1) leads naturally to a particular phase, which for the case of the special relativity is related to the Thomas precession.

III. MATRIX REPRESENTATION

We now explain how the composition law (1) is related to 2×2 matrices.

A. Mathematical definitions

Consider a physical system (see Fig. 1) in which two physical quantities (the inputs) E^+_{in} and E^-_{in} are linearly related

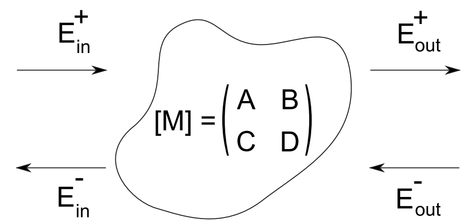

Fig. 1. Schematic representation of the linear relations between input and output quantities.

to two other physical quantities (the outputs), E^+_{out} and E^-_{out}. These relations can be written as a 2×2 matrix as

$$\begin{pmatrix} E^+_{\text{in}} \\ E^-_{\text{in}} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E^+_{\text{out}} \\ E^-_{\text{out}} \end{pmatrix}. \tag{10}$$

This general representation is, for example, used to describe a birefringent system with the help of Jones matrices (see, for example, Ref. 3), or to estimate properties of a multilayer stack with the Abeles matrices.[10] It is possible to define using the 2×2 matrix in Eq. (10), hereafter called $[M]$, the four coefficients $\mathcal{R}^+$, $\mathcal{R}^-$, $\mathcal{T}^+$, and $\mathcal{T}^-$ (this notation is chosen in analogy to the classical reflection and transmission coefficients of a multilayer device),

$$\mathcal{R}^+ = \left.\frac{E^-_{\text{in}}}{E^+_{\text{in}}}\right|_{E^-_{\text{out}}=0} = \frac{C}{A}, \tag{11a}$$

$$\mathcal{R}^- = \left.\frac{E^+_{\text{out}}}{E^-_{\text{out}}}\right|_{E^+_{\text{in}}=0} = -\frac{B}{A}, \tag{11b}$$

$$\mathcal{T}^+ = \left.\frac{E^+_{\text{out}}}{E^+_{\text{in}}}\right|_{E^-_{\text{out}}=0} = \frac{1}{A}, \tag{11c}$$

$$\mathcal{T}^- = \left.\frac{E^-_{\text{in}}}{E^-_{\text{out}}}\right|_{E^+_{\text{in}}=0} = \frac{\det[M]}{A}. \tag{11d}$$

We introduce the variable Θ defined by

$$\Theta = \frac{D}{A}. \tag{12}$$

If we use Eqs. (11) and (12), it is easy to verify that

$$\mathcal{T}^+ \mathcal{T}^- - \mathcal{R}^+ \mathcal{R}^- = \Theta, \tag{13}$$

which constitutes a generalization of the Stokes relation, which is well known in the optics of multilayer devices (see, for example, Ref. 12). We also introduce the conjugation operation (denoted by the bar),

$$\overline{\mathcal{R}^+} = -\mathcal{R}^- = \frac{B}{A} \tag{14a}$$

and

$$\overline{\mathcal{R}^-} = -\mathcal{R}^+. \tag{14b}$$

The conjugation operation does not necessarily correspond to the usual complex conjugation [compare Eqs. (11a) and (14a)]. For Hermitian matrices the correspondence does hold.

With these definitions the $[M]$-matrix can be written as

$$[M]=\frac{1}{\mathcal{T}^{+}}\begin{pmatrix}1 & \overline{\mathcal{R}}^{+}\\ \mathcal{R}^{+} & \Theta\end{pmatrix}. \quad (15)$$

This form of $[M]$ will be useful in the following derivations.

B. Composition laws of the $\mathcal{R}$ and $\mathcal{T}$ variables

We now focus on the properties of the four coefficients $\mathcal{R}_{21}^{+}$, $\mathcal{R}_{21}^{-}$, $\mathcal{T}_{21}^{+}$, and $\mathcal{T}_{21}^{-}$ of a system characterized by its $[M_{21}]$-matrix, resulting from the composition of two subsystems characterized by the two $[M_1]$- and $[M_2]$-matrices defined by

$$[M_1]=\frac{1}{\mathcal{T}_1^{+}}\begin{pmatrix}1 & \overline{\mathcal{R}}_1^{+}\\ \mathcal{R}_1^{+} & \Theta_1\end{pmatrix} \quad \text{and} \quad [M_2]=\frac{1}{\mathcal{T}_2^{+}}\begin{pmatrix}1 & \overline{\mathcal{R}}_2^{+}\\ \mathcal{R}_2^{+} & \Theta_2\end{pmatrix}. \quad (16)$$

The $[M_{21}]$-matrix is the result of the product of two matrices: $[M_{21}]=[M_2][M_1]$. Equations (12)–(14) allow us to express the composition laws for $\mathcal{R}_{21}^{+}$, $\mathcal{T}_{21}^{+}$, $\mathcal{R}_{21}^{-}$, and $\mathcal{T}_{21}^{-}$ as

$$\mathcal{R}_{21}^{+}=\mathcal{R}_2^{+}\oplus\mathcal{R}_1^{+}=\frac{\mathcal{R}_1^{+}\Theta_2+\mathcal{R}_2^{+}}{1+\mathcal{R}_1^{+}\overline{\mathcal{R}}_2^{+}}, \quad (17a)$$

$$\mathcal{R}_{21}^{-}=\mathcal{R}_2^{-}\oplus\mathcal{R}_1^{-}=\frac{\mathcal{R}_1^{-}+\mathcal{R}_2^{-}\Theta_1}{1+\overline{\mathcal{R}}_1^{-}\mathcal{R}_2^{-}}, \quad (17b)$$

$$\mathcal{T}_{21}^{+}=\mathcal{T}_2^{+}\otimes\mathcal{T}_1^{+}=\frac{\mathcal{T}_1^{+}\mathcal{T}_2^{+}}{1+\mathcal{R}_1^{+}\overline{\mathcal{R}}_2^{+}}, \quad (17c)$$

$$\mathcal{T}_{21}^{-}=\mathcal{T}_2^{-}\otimes\mathcal{T}_1^{-}=\frac{\mathcal{T}_1^{-}\mathcal{T}_2^{-}}{1+\overline{\mathcal{R}}_1^{-}\mathcal{R}_2^{-}}. \quad (17d)$$

Equation (14) can be used to show that the denominators in Eq. (17) are the same, $1+\mathcal{R}_1^{+}\overline{\mathcal{R}}_2^{+}=1+\overline{\mathcal{R}}_1^{-}\mathcal{R}_2^{-}$.

C. Composition law of the Θ variables

Although the Θ variable has been introduced in an *ad hoc* way in Eq. (12), it is interesting to find its composition law. We consider two processes characterized by the two variables Θ_1 and Θ_2. If we start from the generalized Stokes relation (13) and use Eq. (17), we find

$$\Theta_{21}=\mathcal{T}_{21}^{+}\mathcal{T}_{21}^{-}-\mathcal{R}_{21}^{+}\mathcal{R}_{21}^{-} \quad (18a)$$

$$=\frac{\mathcal{T}_1^{+}\mathcal{T}_2^{+}\mathcal{T}_1^{-}\mathcal{T}_2^{-}-[\mathcal{R}_1^{+}\Theta_2+\mathcal{R}_2^{+}][\mathcal{R}_1^{-}+\mathcal{R}_2^{-}\Theta_1]}{[1+\mathcal{R}_1^{+}\overline{\mathcal{R}}_2^{+}]^2}. \quad (18b)$$

From Eq. (13) we know that $\mathcal{T}_1^{+}\mathcal{T}_1^{-}=\Theta_1+\mathcal{R}_1^{+}\mathcal{R}_1^{-}$ and $\mathcal{T}_2^{+}\mathcal{T}_2^{-}=\Theta_2+\mathcal{R}_2^{+}\mathcal{R}_2^{-}$. Consequently Eq. (18b) becomes

$$\Theta_{21}=\frac{\Theta_1\Theta_2+\overline{\mathcal{R}}_1^{+}\mathcal{R}_2^{+}}{1+\mathcal{R}_1^{+}\overline{\mathcal{R}}_2^{+}}. \quad (19)$$

This expression can be considered as the composition law for Θ_1 and Θ_2. In Sec. IV we will give the meaning of Θ for various physical contexts.

D. S-matrix

By definition, the four coefficients $\mathcal{R}^{+}$, $\mathcal{T}^{+}$, $\mathcal{R}^{-}$, and $\mathcal{T}^{-}$ are the four elements of the S-matrix associated with scattering,

$$S=\begin{pmatrix}\mathcal{R}^{+} & \mathcal{T}^{-}\\ \mathcal{T}^{+} & \mathcal{R}^{-}\end{pmatrix}. \quad (20)$$

Equation (17) shows that the composition of two S-matrices can be written as

$$S=S_2\circ S_1=\begin{pmatrix}\mathcal{R}_2^{+}\oplus\mathcal{R}_1^{+} & \mathcal{T}_2^{-}\otimes\mathcal{T}_1^{-}\\ \mathcal{T}_2^{+}\otimes\mathcal{T}_1^{+} & \mathcal{R}_2^{-}\oplus\mathcal{R}_1^{-}\end{pmatrix}. \quad (21)$$

The use of the composition laws $\oplus$ and $\otimes$ gives the elements of the S-matrix without resorting to the usual transfer matrices.

IV. THE Θ PHASE FACTOR

We now consider conservative systems described by the unitary matrix $[U]$. In this context the Θ variables are modulus one complex numbers of the form $e^{i\phi}$. Our aim is to show that the phases associated with the physical modes E^{+} and E^{-} in Eq. (10) can be written as a sum of phases when the two modes are not coupled, plus a phase that is simply expressed with the help of the $\oplus$ law.

A. The composition law in the case of unitary matrices

The general expression of a 2×2 unitary matrix is

$$[U]=\begin{pmatrix}\cos\lambda e^{iu} & -\sin\lambda e^{iv}\\ \sin\lambda e^{-iv} & \cos\lambda e^{-iu}\end{pmatrix}e^{i\varphi}, \quad (22)$$

where φ, λ, u, and v are real numbers. The overall phase φ can be omitted without loss of generality and hereafter we set it equal to zero. As we can see, when the modes are not coupled, that is, when $\lambda=0$, the evolution matrix reduces to the simple diagonal expression,

$$[U_{\lambda=0}]=[u]=\begin{pmatrix}e^{iu} & 0\\ 0 & e^{-iu}\end{pmatrix}. \quad (23)$$

The phase difference between the uncoupled modes E^{+} and E^{-} is equal to $2u$. When $\lambda\neq 0$, the evolution of the modes are coupled and the $[U]$-matrix can be factorized as

$$[U]=[U_{\lambda=0}][M]=[u][M] \quad (24a)$$

$$=\begin{pmatrix}e^{iu} & 0\\ 0 & e^{-iu}\end{pmatrix}\begin{pmatrix}\cos\lambda & -\sin\lambda e^{-i(u-v)}\\ \sin\lambda e^{i(u-v)} & \cos\lambda\end{pmatrix}. \quad (24b)$$

Such a factorization will help us to estimate the $\mathcal{R}^{+}$ and Θ components of the different matrices. By using Eqs. (11a), (11c), and (12), we find

$$\mathcal{R}_u^{+}=0, \quad \mathcal{T}_u^{+}=e^{-iu}, \quad \Theta_u=e^{-2iu}, \quad (25)$$

so that

$$[u]=\frac{1}{\mathcal{T}_u^{+}}\begin{pmatrix}1 & 0\\ 0 & \Theta_u\end{pmatrix} \quad (26)$$

and

$$\mathcal{R}_M^+ = \tan\lambda e^{i(u-v)}, \quad \mathcal{T}_M^+ = \frac{1}{\cos\lambda}, \quad \Theta_M = 1. \tag{27}$$

Hence,

$$[M] = \frac{1}{\mathcal{T}_M^+}\begin{pmatrix} 1 & \overline{\mathcal{R}}_M^+ \\ \mathcal{R}_M^+ & 1 \end{pmatrix}. \tag{28}$$

Factorizing the free evolution phases as we did in Eq. (23) will allow us to point out a new phase expressed with the help of the $\oplus$ composition law. For this purpose consider the $[U_{21}]$-matrix, which is the product of two unitary matrices $[U_1]$ and $[U_2]$,

$$[U_{21}] = [U_2][U_1]. \tag{29}$$

The factorization of the free evolution phases gives

$$[U_{21}] = [u_2][M_2][u_1][M_1] \tag{30a}$$

$$=[u_2][u_1][u_1]^{-1}[M_2][u_1][M_1] \tag{30b}$$

$$=[u_2+u_1][M_2(u_1)][M_1] \tag{30c}$$

$$=[u_2+u_1][M_{21}], \tag{30d}$$

where we have defined the diagonal matrix $[u_2+u_1]=[u_2]\times[u_1]$ and noted that $[M_{21}]=[M_2(u_1)][M_1]$ with

$$[M_2(u_1)] = [u_1^{-1}][M_2][u_1] = \begin{pmatrix} \cos\lambda_2 & -\sin\lambda_2 e^{-i(2u_1+u_2-v_2)} \\ \sin\lambda_2 e^{i(2u_1+u_2-v_2)} & \cos\lambda_2 \end{pmatrix}. \tag{31}$$

If we use definition (12), we easily find

$$\Theta_{M_1} = \Theta_{M_2} = \Theta_{M_2(u_1)} = 1. \tag{32}$$

From the composition law (19), we obtain

$$\Theta_{M_{21}} = \frac{\Theta_{M_1}\Theta_{M_2(u_1)} + \mathcal{R}^+_{M_2(u_1)}\overline{\mathcal{R}}^+_{M_1}}{1+\overline{\mathcal{R}}^+_{M_2(u_1)}\mathcal{R}^+_{M_1}} = \frac{1+\overline{\mathcal{R}}^+_{M_1}\mathcal{R}^+_{M_2(u_1)}}{1+\mathcal{R}^+_{M_1}\overline{\mathcal{R}}^+_{M_2(u_1)}}, \tag{33}$$

or using the composition law definition in Eq. (1),

$$\Theta_{M_{21}} = \frac{\mathcal{R}^+_{M_2(u_1)} \oplus \mathcal{R}^+_{M_1}}{\mathcal{R}^+_{M_1} \oplus \mathcal{R}^+_{M_2(u_1)}}. \tag{34}$$

It is interesting to note that $\Theta_{M_{21}}$ comes from the noncommutativity of the composition law $\oplus$. Although distinct, the two composite quantities $\mathcal{R}^+_{M_1}\oplus\mathcal{R}^+_{M_2(u_1)}$ and $\mathcal{R}^+_{M_2(u_1)}\oplus\mathcal{R}^+_{M_1}$ have the same modulus, so that $\Theta_{M_{21}}$ is a pure phase term

$$\Theta_{M_{21}} = e^{-2i\phi}. \tag{35}$$

Finally, the whole phase term $\Theta_{U_{21}}=e^{-2i\phi_{21}}$ associated with the $[U_{21}]$-matrix is

$$\Theta_{U_{21}} = e^{-2i\phi_{12}} = \Theta_{u_1+u_2}\Theta_{M_{21}} = \Theta_{M_{21}}e^{-2i(u_1+u_2)}, \tag{36}$$

which gives the phase

$$\phi_{21} = u_1 + u_2 + \phi. \tag{37}$$

The noncommutativity of the $\oplus$ law implies $\Theta_{M_{21}} \neq 1$ in Eq. (34) and is responsible for the additional phase ϕ appearing in Eq. (37).

B. Examples of the physical meaning of the Θ variable

In the following we illustrate the meaning of the phase term Θ by three examples from different fields of physics.

1. *Special relativity*

We first choose the composition of two nonparallel velocities $\vec{v_1}$ and $\vec{v_2}$. In this case the four elements A, B, C, and D of matrix (10) are respectively $\cosh(a_i/2)$, $\sinh(a_i/2)e^{-i\alpha_i}$, $\sinh(a_i/2)e^{i\alpha_i}$, and $\cosh a_i/2$, where a_i and $v_i=\tanh a_i$ are respectively the rapidity and the velocity of the reference frame i for a given observer. Equations (11a) and (14a) then give $V_i=\mathcal{R}_i^+=\tanh(a_i/2)e^{i\alpha_i}$ and $\bar{V}_i=\overline{\mathcal{R}}_i^+=\tanh(a_i/2)e^{-i\alpha_i}$. Here phase α_i gives the orientation of $\vec{v_i}$ with respect to an arbitrary axis belonging to the plane defined by the vectors $\vec{v_1}$ and $\vec{v_2}$ in the observer reference frame. Because $\Theta_i=1$, Eqs. (19), (33), and (34) give

$$\Theta_{21} = \frac{1+\overline{\mathcal{R}}_1^+\mathcal{R}_2^+}{1+\mathcal{R}_1^+\overline{\mathcal{R}}_2^+} = \frac{1+\bar{V}_1V_2}{1+V_1\bar{V}_2} = \frac{1+\tanh\frac{a_1}{2}e^{-i\alpha_1}\tanh\frac{a_2}{2}e^{i\alpha_2}}{1+\tanh\frac{a_1}{2}e^{i\alpha_1}\tanh\frac{a_2}{2}e^{-i\alpha_2}}. \tag{38}$$

This expression is a pure phase term and can be written as

$$\Theta_{21} = e^{-2i\phi}, \tag{39}$$

where 2ϕ is Wigner's angle associated with the Thomas precession. Note that from Eq. (38) we directly obtain the value of the Wigner angle. The real part of Θ_{21} gives immediately the known result,[1,11]

$$\cos 2\phi = \frac{\left(1+\tanh\frac{a_1}{2}\tanh\frac{a_2}{2}\cos\alpha\right)^2 - \left(\tanh\frac{a_1}{2}\tanh\frac{a_2}{2}\sin\alpha\right)^2}{\left(1+\tanh\frac{a_1}{2}\tanh\frac{a_2}{2}\cos\alpha\right)^2 + \left(\tanh\frac{a_1}{2}\tanh\frac{a_2}{2}\sin\alpha\right)^2}, \tag{40}$$

where $\alpha=\alpha_2-\alpha_1$.

As was shown at the end of Sec. IV A, the Wigner angle comes from the noncommutativity of the composition law, which mimics the noncommutativity of the Lorentz boosts. The $\oplus$ law can be easily used for the composition of any number of coplanar velocities. For example, for three referential frames, we obtain by iterating Eq. (1),

$$W = V_3 \oplus (V_2 \oplus V_1) = \frac{V_1+V_2+V_3+V_1\bar{V}_2V_3}{1+\bar{V}_1V_2+\bar{V}_1V_3+\bar{V}_2V_3}. \tag{41}$$

2. The optics of stratified media

This example is from the optics of stratified media. If r_i and t_i are the Fresnel reflection and transmission coefficients of interface i $(i=1,2)$, and ϕ_i is the phase shift associated with the propagation of light, the four elements A, B, C, and D of matrix (10) are respectively $1/t_i$, $(r_i/t_i)e^{-i\phi_i}$, $(r_i/t_i)e^{i\phi_i}$, and $1/t_i$. Equation (11a) gives $\mathcal{R}_i^+=r_i e^{i\phi_i}$, so that Eq. (17a) gives the overall reflection coefficient[12] of the two interfaces (6). In this case a phase term[13] also appears, which is strictly similar to the Wigner angle in special relativity. Its origin comes also from the noncommutativity of the $\oplus$ law, which is related to the noninvariance of the problem when the two interfaces are exchanged.

3. Light wave polarization

We consider two nonperfect polarizers as in our last example. The quality of the polarizer resulting from using successively two polarizers P_1 and P_2 is given[3] by Eq. (1). It is interesting to calculate the value of Θ_{21} in this case. The noncommutativity of the two quantities that are *composed* is expressed by Θ_{21}. In special relativity finding two different results when calculating the resulting velocity of v_1 composed with v_2 and of v_2 composed with v_1 might have been surprising. It is not the case with polarizers. It is well known that the final polarization of a light wave going through the polarizer P_1 and then through the polarizer P_2 is not the same as the final polarization of the light wave going first through polarizer P_2 and then through polarizer P_1. We consider explicitly the noncommutativity of polarizers. From Eqs. (8) and (19) we obtain

$$e^{-2i\Omega}=\frac{1+\tanh\dfrac{\gamma_1}{2}e^{i\alpha_1}\tanh\dfrac{\gamma_2}{2}e^{-i\alpha_2}}{1+\tanh\dfrac{\gamma_1}{2}e^{-i\alpha_1}\tanh\dfrac{\gamma_2}{2}e^{i\alpha_2}}. \tag{42}$$

In Eq. (42) 2Ω is the angle between the polarization of light $\vec{E}_{12}$ when going through the two polarizers in the order P_1 and then P_2 and that of light $\vec{E}_{21}$ when going through the polarizers in the order P_2 and P_1. For two perfect polarizers we expect to find $2\Omega=\alpha_2-\alpha_1=\alpha$. To verify this result, replace $\tanh(\gamma_1/2)$ and $\tanh(\gamma_2/2)$ by unity for perfect polarizers, and then the corresponding Eq. (42) for polarizers gives the expected result

$$\cos 2\Omega=\cos\alpha. \tag{43}$$

V. DISCUSSION

From Eq. (28), we observe that there are redundancies of information in 2×2 unitary matrices. All information is contained in the first (or the second) column. Because of this redundancy, it is easy to understand why the use of the composition law (1) is easier and more rapid than using matrix methods such as transfer matrices. Moreover, as shown in Ref. 4, calculations converge more rapidly when the composition law is used. This rapid convergence comes from the fact that the denominator of the composition law is a normalization factor. Another useful aspect of the composition law is that it can be easily iterated as

$$\mathcal{R}^+_{n,\ldots,1}=\mathcal{R}^+_n\oplus(\mathcal{R}^+_{n-1}\oplus\cdots(\mathcal{R}^+_2\oplus\mathcal{R}^+_1)). \tag{44}$$

As mentioned, this property leads to efficient algorithms for many kinds of problems. Also, Eq. (44) is so simple that its analytic value can be directly given without any matrix calculations. Equation (44) is a complex generalization of the elementary symmetric functions of the mathematical theory of polynomials:[14] the numerator of $\mathcal{R}^+_{n,\ldots,1}$ is constituted by all the possible odd ordered products of the different $\mathcal{R}_i^+$ factors such that in each product, the $\mathcal{R}^+$ and $\overline{\mathcal{R}}^+$ factors appear alternatively, the first factor always being $\mathcal{R}^+$. The denominator of $\mathcal{R}^+_{n,\ldots,1}$ is constituted by all the possible even ordered products of $\mathcal{R}_i^+$, such that in each product, the $\overline{\mathcal{R}}^+$ and $\mathcal{R}^+$ factors appear alternatively, the first always being $\overline{\mathcal{R}}^+$. If we limit ourselves to two iterations, the value of $\mathcal{R}^+_{3,\ldots,1}$ is directly given by

$$\mathcal{R}^+_{3,\ldots,1}=\mathcal{R}^+_3\oplus(\mathcal{R}^+_2\oplus\mathcal{R}^+_1) \tag{45a}$$

$$=\frac{\mathcal{R}^+_1+\mathcal{R}^+_2+\mathcal{R}^+_3+\mathcal{R}^+_1\overline{\mathcal{R}}^+_2\mathcal{R}^+_3}{1+\overline{\mathcal{R}}^+_1\mathcal{R}^+_2+\overline{\mathcal{R}}^+_1\mathcal{R}^+_3+\overline{\mathcal{R}}^+_2\mathcal{R}^+_3}. \tag{45b}$$

Such a result can be useful for the case of S-matrices because the generalization of Eq. (45) allows us to write the S-matrix simply as

$$S=\begin{pmatrix}\mathcal{R}^+_n\oplus\cdots(\mathcal{R}^+_2\oplus\mathcal{R}^+_1) & \mathcal{T}^-_n\otimes\cdots(\mathcal{T}^-_2\otimes\mathcal{T}^-_1)\\ \mathcal{T}^+_n\otimes\cdots(\mathcal{T}^+_2\otimes\mathcal{T}^+_1) & \mathcal{R}^-_n\oplus\cdots(\mathcal{R}^-_2\oplus\mathcal{R}^-_1)\end{pmatrix}. \tag{46}$$

It is well known that a number of physical processes are more adequately described by S-matrices than by T-matrices (transfer matrices). Unfortunately, whereas T-matrices must be successively multiplied together, $[M_{n,\ldots,1}]=[M_n]\times[M_{n-1}]\cdots[M_2][M_1]$, such is not the case with S-matrices. The composition law is consequently useful for S-matrices because our results show how to directly calculate the four elements of the overall S-matrix by iterating Eq. (17).

VI. CONCLUSION

The composition law of velocities in special relativity appears to be the natural way to add velocities that are subject to the condition $|v|<c$. Its generalization in the complex plane leads to simple calculations of bounded quantities which would be otherwise difficult to calculate. We have shown how, for example, the Wigner angle in special relativity, the overall reflection coefficient of any multilayer, and the effect of any number of polarizers can be directly obtained from this general composition law. Also, we have shown that the generalization of Einstein's composition law provides a natural way to compose scattering matrices.

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Gunpowder Building. Setting off the gunpowder bomb, shown inside the lithographed tin building, causes it to collapse, and wakes up the students in the back row of the classroom. The apparatus is in the Moosenick Museum at Transylvania University in Lexington, Kentucky. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)