

Composition law for polarizersJ. Lages^{*}*Institut UTINAM, UMR CNRS 6213, Equipe de Dynamique des Structures Complexes, Université de Franche-Comté, UFR ST, Route de Gray, 25030 Besançon Cedex, France*R. Giust[†]*FEMTO-ST, UMR CNRS 6174, Université de Franche-Comté, UFR ST, Route de Gray, 25030 Besançon Cedex France*J.-M. Vigoureux[‡]*Institut UTINAM, UMR CNRS 6213, Equipe de Dynamique des Structures Complexes, Université de Franche-Comté, UFR ST, 16 Route de Gray, 25030 Besançon Cedex, France*

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The polarization process when polarizers act on an optical field is studied. We give examples for two kinds of polarizers. The first kind presents an anisotropic absorption—as in a Polaroid film—and the second one is based on total reflection at the interface with a birefringent medium. Using the Stokes vector representation, we determine explicitly the trajectories of the wave light polarization during the polarization process. We find that such trajectories are not always geodesics of the Poincaré sphere as is usually thought. Using the analogy between light polarization and special relativity, we find that the action of successive polarizers on the light wave polarization is equivalent to the action of a single resulting polarizer followed by a rotation achieved, for example, by a device with optical activity. We find a composition law for polarizers similar to the composition law for noncollinear velocities in special relativity. We define an angle equivalent to the relativistic Wigner angle which can be used to quantify the quality of two composed polarizers.

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I. INTRODUCTION

It is now usual to describe the polarization state of an electromagnetic field by using the notion of the Stokes vector and that of the Poincaré sphere. This elegant representation of the polarization state is very useful in optics; e.g., geometrical phases such as the Pancharatnam phase can be efficiently interpreted on the Poincaré sphere [1,2]. Actions of polarizing devices are described in such a space with simple mathematical operations. As an example, variations of the polarization state when light passes through an optical active medium are expressed by a simple rotation around the axis connecting the two poles of the sphere. In the same way, the effect of a birefringent plate corresponds to a rotation around a vector lying in the equatorial plane by an angle determined by the optical path delay between the ordinary and extraordinary axes of the plate. However, it is curiously interesting to note that the evolution of the polarization due to the action of a polarizer has never been studied in detail. On the Poincaré sphere, such an evolution corresponds to a trajectory intuitively supposed to be a geodesic connecting the polarization states of the field before and after the polarizer (see, e.g., [4,5]). As we will see, the study of the actions of two different kinds of polarizers (the first one using an anisotropic absorbing medium and the second one using total reflection of one component of the electromagnetic field at the interface with an anisotropic media) will show that such

an assumption is not always true and that the trajectories of the polarization state of the field can be more complex and do not necessarily correspond to Poincaré sphere geodesics. At the beginning of the article, we introduce the Jones representation of a polarizer. The action of a polarizer on the polarization state is then studied by introducing rotation operators and Lorentz boosts operators borrowed from special relativity. Since the $SL(2, \mathbb{C})$ group is homomorphic to the Lorentz group (see, e.g., [6,7]), a formal equivalence between the special relativity space and the polarization space exists [8,9]. This formal equivalence can then be used to determine what the counterparts are in the polarization space of, for example, the composition law of velocities, of the aberration phenomenon, or of the Wigner angle.

The paper is organized as follows. In Sec. II we define the mathematical formalism used to describe the effect of polarizers on an optical field polarization. The states of polarization of the field are represented on the Poincaré sphere and the action of the optical devices is defined by their associated Jones matrix. It appears that the effect of polarizers on the polarization is essentially defined by a complex number A which contains information about the propagation and the absorption of light in the polarizer. In Sec. III we analyze the evolution of the Stokes vector \vec{s} when light goes through different kinds of polarizers. Examples are studied, and the trajectories of given optical states during the polarization process are analyzed. We point out that the projection of such trajectories on the Poincaré sphere is no longer a geodesic of that sphere as could have been classically expected. In Sec. IV we focus on the evolution of the optical field intensity and we derive a natural generalized Malus law. Then we examine the action of two successive polarizers and we show that they are equivalent to one polarizer followed by a rotation exactly

^{*}jose.lages@utinam.cnrs.fr[†]remo.giust@univ-fcomte.fr[‡]jean-marie.vigoureux@univ-fcomte.fr

as in special relativity where the composition of two Lorentz boosts is a Lorentz boost followed by a rotation. In Sec. V we find the general composition law which permits one to determine characteristics of the equivalent polarizer and the angle of the rotation. Then we discuss the interpretation of such a composition law. In Sec. VI we give the counterpart of the Wigner angle in special relativity for two composed polarizers and we relate such an angle to the quality of the polarizers.

II. MATHEMATICAL DESCRIPTION

A. Stokes parameters and the Poincaré sphere

The polarization state of light will be described with the use of the Poincaré sphere. Our goal is to describe the evolution of the polarization state during its propagation inside a polarizer. Without loss of generality we consider that the optical field propagates along the z direction. This field can then be described by the Jones vector

$$|\tilde{\psi}\rangle = \begin{pmatrix} E_x \\ E_y \end{pmatrix}_{\{|x\rangle, |y\rangle\}}, \quad (1)$$

where E_x and E_y are the two complex components of the electric field. Instead of the basis $\{|x\rangle, |y\rangle\}$, we prefer to work with the basis $\{|R\rangle, |L\rangle\}$ related to right- and left-handed circularly polarized states:

$$|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}_{\{|x\rangle, |y\rangle\}}, \quad |L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}_{\{|x\rangle, |y\rangle\}}. \quad (2)$$

In this basis the Jones vector $|\tilde{\psi}\rangle$, (1), is transformed into [3]

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} |\tilde{\psi}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} E_x - iE_y \\ E_x + iE_y \end{pmatrix}_{\{|R\rangle, |L\rangle\}}. \quad (3)$$

The intensity I and the polarization of the wave light can be characterized by the Stokes parameters [10]

$$\begin{aligned} s^0 &= |E_x|^2 + |E_y|^2 \equiv I, \\ s^1 &= |E_x|^2 - |E_y|^2, \\ s^2 &= 2 \operatorname{Re}(E_x^* E_y), \\ s^3 &= 2 \operatorname{Im}(E_x^* E_y). \end{aligned} \quad (4)$$

If we construct a three-dimensional vector \vec{s} (called hereafter the Stokes vector) with the components s^1 , s^2 , and s^3 , it is easy to check that its norm $s = \|\vec{s}\| = s^0 = I$. Thus, for a given intensity, the Stokes vector can be parametrized using spherical coordinates as

$$\vec{s} = s[(\sin \theta \cos \phi) \vec{e}^1 + (\sin \theta \sin \phi) \vec{e}^2 + (\cos \theta) \vec{e}^3], \quad (5)$$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi[$, and $\{\vec{e}^1, \vec{e}^2, \vec{e}^3\}$ is an \mathbb{R}^3 orthonormal basis. The Stokes vector \vec{s} , (5), determines a point located on the Poincaré sphere \mathcal{S}^2 of radius s . Since the norm s gives the intensity of the wave light, the direction of the vector \vec{s} —i.e., the angles θ and ϕ —characterizes the polar-

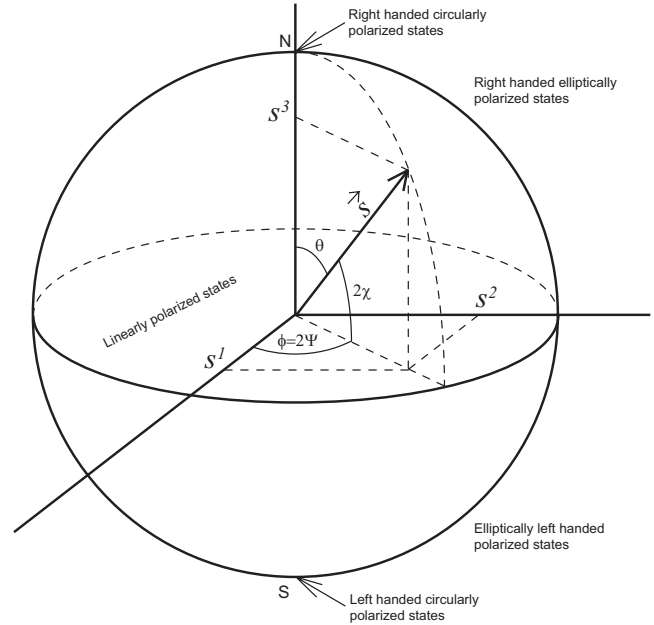


FIG. 1. Representation of the Stokes vector \vec{s} and the Poincaré sphere \mathcal{S}^2 .

ization. Thus, the polarization state of the light wave corresponds to a unique point on the Poincaré sphere \mathcal{S}^2 (see Fig. 1).

In order to follow the evolution of the light wave polarization—i.e., the evolution of the Stokes vector \vec{s} —we prefer to work with the projector $|\psi\rangle\langle\psi|$ instead of $|\psi\rangle$ itself. Indeed, this projector can be easily expressed as a function of \vec{s} since

$$|\psi\rangle\langle\psi| = \frac{1}{2} \begin{pmatrix} s^0 + s^3 & s^1 - is^2 \\ s^1 + is^2 & s^0 - s^3 \end{pmatrix} = \frac{1}{2} (s^0 \sigma^0 + \vec{s} \cdot \vec{\sigma}) \equiv \rho_{\vec{s}}. \quad (6)$$

Here σ^0 is the 2×2 identity matrix and $\vec{\sigma}$ is a vector the components of which are the usual Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7)$$

Usually [10], one chooses as the north (south) pole of the Poincaré sphere the circular right-handed (left-handed) polarization state. With this definition, the angles θ and ϕ , (5), are directly related with the ellipticity angle χ and the azimuth angle Ψ since $\theta = \pi/2 - 2\chi$ and $\phi = 2\Psi$, respectively [10]. The circular right- (left-) handed polarization state corresponds to $\theta = 0$ ($\theta = \pi$). Right- (left-) handed elliptically polarized states correspond to $s^3 > 0$ ($s^3 < 0$) or, equivalently, $\chi > 0$ ($\chi < 0$). Linear polarization states are represented by Stokes vectors lying in the equatorial plane of the Poincaré sphere since $\theta = \pi/2$ ($\chi = 0$). Each linear polarization state is determined by a given angle ϕ or, equivalently, by a given azimuth angle Ψ . For a linear polarized wave, the angle Ψ is defined as the angle between the vibration plane and a laboratory axis \vec{e}_x (reference axis) orthogonal to the direction of the propagation. The difference of polarization for two linear polarized light waves 1 and 2 can be measured, in the laboratory, as the difference $\Delta\Psi = \Psi_1 - \Psi_2$ or, equivalently, on the

Poincaré sphere, as $\Delta\phi = \phi_1 - \phi_2 = 2\Delta\Psi$. Thus an angle measured in the laboratory is half of the corresponding angle in the Poincaré sphere representation. For example, two orthogonal polarizations are characterized by $\Delta\Psi = \pi/2$ in the laboratory frame and by $\Delta\phi = \pi$ in the Poincaré sphere representation. Although the rest of the paper does not consider any particular choice of basis or any particular choice of polarization, it is useful to keep in mind these last remarks in order to be able to make at any time the parallel case with known conventions in optics. So, in a general manner, two states with orthogonal polarizations are represented by opposite Stokes vectors and correspond to projectors $\rho_{\vec{s}}$ and $\rho_{-\vec{s}}$.

B. Jones matrices for polarizing devices

1. Jones matrix for birefringent systems

The action of any optical system on the light wave state $|\tilde{\psi}\rangle$ (1) is defined by a 2×2 Jones matrix. For example, the Jones matrix associated with a birefringent system and acting on $|\tilde{\psi}\rangle$ can be written as

$$\begin{pmatrix} e^{i\phi_o} & 0 \\ 0 & e^{i\phi_e} \end{pmatrix} = e^{i(\phi_o + \phi_e)/2} \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \quad (8)$$

Here ϕ_o and ϕ_e are the ordinary and extraordinary phases associated with the propagation of the optical field polarized along each optical axis. We have defined the difference between these two phases as $\phi = \phi_o - \phi_e$. For ordinary and extraordinary axes rotated by an angle θ in the (x, y) laboratory frame, the Jones matrix (8) becomes

$$\begin{aligned} \tilde{B}(\theta, \phi_o, \phi_e) &= e^{i(\phi_o + \phi_e)/2} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \\ &\times \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \end{aligned} \quad (9)$$

The corresponding Jones matrix for a birefringent optical device acting now on the Jones vector $|\psi\rangle$ written in the $\{|R\rangle, |L\rangle\}$ basis (3) is then

$$B(\theta, \phi_o, \phi_e) = \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \tilde{B}(\theta, \phi_o, \phi_e) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \quad (10)$$

After performing the matrix multiplications, it is possible to recast (10) in terms of Pauli matrices:

$$B(\theta, \phi_o, \phi_e) = e^{i(\phi_o + \phi_e)/2} \left(\cos \frac{\phi}{2} \sigma^0 + i \sin \frac{\phi}{2} \vec{p} \cdot \vec{\sigma} \right), \quad (11)$$

where $\vec{p} = (\cos 2\theta)\vec{e}^1 + (\sin 2\theta)\vec{e}^2$. Using now the mathematical relation $e^{i\alpha\vec{q} \cdot \vec{\sigma}} = \cos \alpha + i \sin \alpha \vec{q} \cdot \vec{\sigma}$ valid for any angle α and any unitary vector \vec{q} , we can rewrite (11) in a more compact manner

$$B(\theta, \phi_o, \phi_e) = e^{i\phi_o} e^{-i\phi/2} e^{i\phi/2 \vec{p} \cdot \vec{\sigma}}. \quad (12)$$

As we will see in Sec. III A, the action of the matrix $B(\theta, \phi_o, \phi_e)$ on the polarization state $|\psi\rangle$ can be viewed in the Poincaré sphere framework as the rotation of the Stokes vector \vec{s} around the vector \vec{p} by an angle ϕ . As no dissipation

phenomenon occurs, the norm s^0 of the vector \vec{s} is constant.

2. Jones matrix for polarizers

The Jones matrix associated with a perfect polarizer acting on $|\tilde{\psi}\rangle$ can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (13)$$

In that example only the x component of the field is conserved; the orthogonal component is not transmitted. This oversimplified picture corresponds to the limit case where the attenuation factors in the x and y directions differ from each other by several orders of magnitude. The realistic case corresponds then to the matrix

$$\begin{pmatrix} e^{-\gamma_1} & 0 \\ 0 & e^{-\gamma_2} \end{pmatrix} = e^{-(\gamma_1 + \gamma_2)/2} \begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2} \end{pmatrix}, \quad (14)$$

where $e^{-\gamma_1}$ and $e^{-\gamma_2}$ are the attenuation factors in the x and y directions, respectively. We have defined the difference between the two attenuation terms as $\gamma = \gamma_2 - \gamma_1$. Again, if the polarizer axis is rotated about an angle θ in the (x, y) laboratory frame, the Jones matrix (14) becomes

$$\begin{aligned} \tilde{P}(\theta, \gamma_1, \gamma_2) &= e^{-(\gamma_1 + \gamma_2)/2} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix} \\ &\times \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \end{aligned} \quad (15)$$

Following the same steps as in the previous section for birefringent systems, the Jones matrix for a polarizer acting on $|\psi\rangle$ in the $\{|R\rangle, |L\rangle\}$ basis (3) is

$$P(\theta, \gamma_1, \gamma) = e^{-\gamma_1} e^{-\gamma/2} e^{\gamma/2 \vec{p} \cdot \vec{\sigma}}, \quad (16)$$

where the vector \vec{p} encodes again information about the direction of the polarizer in the laboratory frame. In the Poincaré sphere framework, as the attenuation difference γ increases, the action of the operator (16) on $|\psi\rangle$ can be seen as the progressive collapse of the Stokes vector \vec{s} on the polarizer vector \vec{p} direction (see Sec. III B).

We now use the derived expressions (12) and (16) to examine more precisely the case of a polarizer based on anisotropic absorption (such as a Polaroid film) and the case of a polarizer based on a total reflection at the interface of an anisotropic crystal.

The case of an anisotropic absorbing polarizer can be modeled by a system where the phases ϕ_o and ϕ_e in orthogonal directions are complex:

$$\begin{aligned} \phi_o &= (k + i\alpha_o)z, \\ \phi_e &= (k + i\alpha_e)z. \end{aligned} \quad (17)$$

The parameter k is the wave vector of the optical field in the film, and α_o and α_e are the two absorption coefficients (for the amplitude) in two orthogonal directions of polarization. The system acts as a linear polarizer if $\alpha_o \neq \alpha_e$. In (17), the parameter z is the propagation coordinate; i.e., $z=0$ at the entrance of the polarizer and, e.g., $z=L$ at its end. Hence,

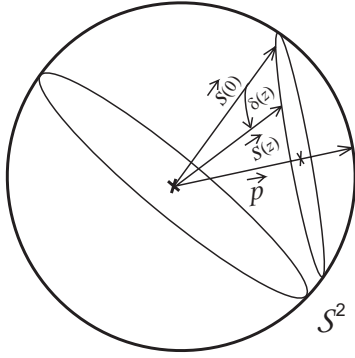


FIG. 2. In the case of a nonabsorbing polarizing device, the Stokes vector \vec{s} is rotated around the polarization vector \vec{p} .

using (12), the operator acting on $|\psi\rangle$ associated to such a polarizer is

$$P_1(\theta, \phi_o, \gamma) = e^{ikz} e^{-\alpha_o z} e^{-\gamma/2} e^{\gamma/2 \vec{p} \cdot \vec{\sigma}}, \quad (18)$$

where $\gamma = (\alpha_e - \alpha_o)z$. Up to a global phase term and a global attenuation term which do not affect the light wave polarization, we retrieved, as expected, the expression of the Jones matrix for a polarizer, (16).

The other case corresponds to a polarizer based on total reflection. We consider a polarized light hitting the face of an anisotropic crystal in such a manner that only one component of the polarized light is transmitted, the other component being evanescent after the interface. This kind of polarizer can be modeled by an optical device with the phases ϕ_o and ϕ_e defined as

$$\begin{aligned} \phi_o &= kz, \\ \phi_e &= i\mu z, \end{aligned} \quad (19)$$

where k and μ are the z real components of the wave vector of the linear polarization which are transmitted and totally reflected, respectively. Here, the parameter z is the propagation coordinate along the transmitted light wave component ($z=0$ at the interface). Again, using (12), the Jones operator acting on the wave light $|\psi\rangle$ and associated to such polarizer is

$$P_2(\theta, \delta, \gamma) = e^{ikz} e^{-(\gamma+i\delta)/2} e^{(\gamma+i\delta)/2 \vec{p} \cdot \vec{\sigma}}, \quad (20)$$

where $\gamma = \mu z$ and $\delta = kz$.

Comparing the just derived expressions (12), (16), (18), and (20) of Jones operators (Jones matrices) for polarizing devices, we remark that all of them are written in the following manner:

$$P(\vec{p}, \phi_p, A) = e^{i\phi_p} e^{-A/2} e^{A/2 \vec{p} \cdot \vec{\sigma}}, \quad (21)$$

where A is a complex differential attenuation term, ϕ_p an overall phase and attenuation term, and \vec{p} the polarization vector associated with the axis of the polarizer. In the following we will drop out the overall factor $e^{i\phi_p}$ from the expression of the Jones operator, (21), since it does not affect the evolution of the wave light polarization. We have just to keep in mind that in the case of linear polarizers such as those defined in and (16) and (18), the overall factor $e^{i\phi_p}$

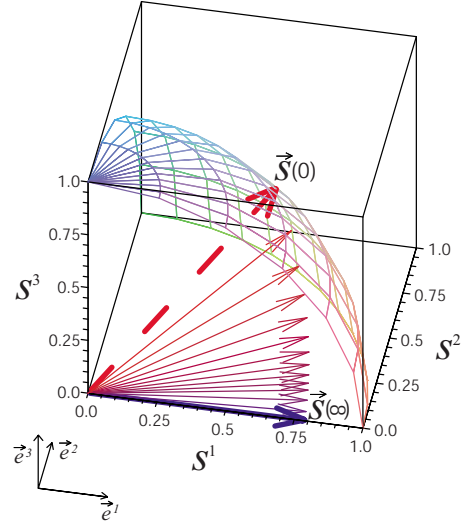


FIG. 3. (Color online) Transformation (31) of the spatial components $\vec{s}(z)$ of the Stokes four-vector $\mathbf{s}(z)$. The polarization vector is $\vec{p} = \vec{e}^1$. The incoming light wave has an initial polarization $\vec{s}(0) = (1/\sqrt{3})(\vec{e}^1 + \vec{e}^2 + \vec{e}^3)$ represented by the thick red (light gray) dashed vector. The Stokes vector for $\gamma \rightarrow \infty$ is aligned along the polarization vector $\vec{p} = \vec{e}^1$ and is represented by the thick blue (dark gray) vector. The other vectors are drawn to show the intermediate positions of the Stokes vector \vec{s} as the parameter γ (or equivalently the parameter z) increases from 0 to ∞ .

contains an attenuation part which globally reduces the intensity of the wave light. From now on, it will be useful to split A into its real and imaginary parts, $A = \gamma + i\delta$. Note that A and, consequently, γ and δ are implicit linear functions of the z propagation coordinate.

Also, although (21) has been derived from the analysis of linear polarizers (vector \vec{p} lying in the equatorial plane), our model is completely general. It is possible to use a more general rotation than (15), bringing the polarizer characteristic vector \vec{p} outside the equatorial plane. For example, if the vector \vec{p} points toward the Poincaré sphere north (or south) pole, our model (21) describes circular polarizers or, if $A = i\delta$, media with optical activity.

III. LIGHT WAVE POLARIZATION VIEWED AS A STOKES FOUR-VECTOR TRANSFORMATION

According to the previous section, the action of a polarizing device on a light wave can be characterized by the following operator:

$$P_{\vec{p}, \gamma, \delta}(z) = e^{-(\gamma+i\delta)/2} e^{(\gamma+i\delta)/2 \vec{p} \cdot \vec{\sigma}}, \quad (22)$$

where $\gamma(z)$ and $\delta(z)$ are real positive functions of the transformation parameter $z \in \mathbb{R}^+$. Usually, z is the length penetration of light into the device and the functions $\gamma(z)$ and $\delta(z)$ encode absorption and propagation of light inside the device, respectively. We restrict our work to homogeneous media for which these functions are linear in z ; i.e., $\gamma(z) = \alpha z$ and $\delta(z) = \beta z$, with $\alpha, \beta \in \mathbb{R}$. The vector \vec{p} is a normalized polarization vector ($\|\vec{p}\| = p = 1$) associated with the polarizing device. The vector \vec{p} determines a point on the Poincaré sphere

which corresponds to a pure state $\rho_{\vec{p}}$ of polarization \vec{p} .

Using the projector $\rho_{\vec{s}}=|\psi\rangle\langle\psi|$ we are able to define the Stokes four-vector $\mathbf{s}=(s^0,\vec{s})$ with the following components:

$$s^\mu = \text{Tr}(\sigma^\mu \rho_{\vec{s}}), \quad \mu \in \{0, 1, 2, 3\}. \quad (23)$$

It is interesting to note that, introducing the Minkowskian metric $\eta_{\mu\nu}=\text{diag}\{-1, 1, 1, 1\}$, the Stokes four-vector can be considered as a *lightlike* four-vector since $\eta_{\mu\nu}s^\mu s^\nu=0$. The spatial components of the Stokes four-vector, $\{s^i\}_{i=1,2,3}$, are the components of the three-dimensional Stokes vector \vec{s} defined in the previous section. The *time* component of the Stokes four-vector s^0 is the norm of the Stokes vector \vec{s} . Consequently this last component also corresponds to the light wave intensity $s^0=\langle\psi|\psi\rangle=I$.

So, according to the operator (22), a polarization state $\rho_{\vec{s}(0)}$ is continuously transformed into another polarization state $\rho_{\vec{s}(z)}$ by the following operation:

$$\rho_{\vec{s}(z)} = P_{\vec{p},\gamma,\delta}(z)\rho_{\vec{s}(0)}P_{\vec{p},\gamma,\delta}^\dagger(z), \quad z \in \mathbb{R}^+. \quad (24)$$

Now, using (23) and (24) we are able to define the evolution of the Stokes four-vector $\mathbf{s}(z)$ by

$$s^\mu(z) = \text{Tr}(\sigma^\mu \rho_{\vec{s}(z)}) = \text{Tr}(P_{\vec{p},\gamma,\delta}^\dagger(z)\sigma^\mu P_{\vec{p},\gamma,\delta}(z)\rho_{\vec{s}(0)}), \quad \mu \in \{0, 1, 2, 3\}, \quad (25)$$

where we recall that $s^0(z)=s(z)=\|\vec{s}(z)\|=I(z)$ is the intensity of the light wave and $\{s^i(z)\}_{i\in\{1,2,3\}}$ are the spatial components of the Stokes vector $\vec{s}(z)$. After some algebraic calculus, the components $s^\mu(z)$ of the Stokes four-vector (25) can be expressed as

$$s^\mu(z) = \begin{cases} s^0(z) = e^{-\gamma(z)}[s^0(0)\cosh \gamma(z) + \sinh \gamma(z)\vec{s}(0) \cdot \vec{p}], \\ \vec{s}(z) = e^{-\gamma(z)}\{\vec{s}(0)\cos \delta(z) + \vec{s}(0) \times \vec{p} \sin \delta(z) + [1 - \cos \delta(z)][\vec{p} \cdot \vec{s}(0)]\vec{p} + s^0(0)\vec{p} \sinh \gamma(z) + [\cosh \gamma(z) - 1][\vec{p} \cdot \vec{s}(0)]\vec{p}\}. \end{cases} \quad (26)$$

For the sake of clarity we now consider the two particular cases $\gamma=0, \delta \neq 0$ and $\gamma \neq 0, \delta=0$ before considering the more general case of polarizers with $\gamma \neq 0$ and $\delta \neq 0$.

A. Nonabsorbing polarizing device: $\gamma=0, \delta \neq 0$

The unitary case $\gamma=0, \delta \neq 0, \forall z \in \mathbb{R}$ corresponds to nonabsorbing birefringent devices or to media with optical activity. In such a case, the transformation (26) of the Stokes four-vector \mathbf{s} becomes

$$s^\mu(z) = \begin{cases} s^0(z) = s^0(0) \equiv 1, \\ \vec{s}(z) = \vec{s}(0)\cos \delta(z) + \vec{s}(0) \times \vec{p} \sin \delta(z) + [1 - \cos \delta(z)][\vec{p} \cdot \vec{s}(0)]\vec{p}. \end{cases} \quad (27)$$

The first equation in (27) verifies that $s(z)=\|\vec{s}(z)\|=s^0(0) \equiv 1$ is invariant under the action of the nonabsorbing polarizing device. So the light wave intensity I is conserved and for convenience we have set $I \equiv 1$. The extremity of the vector $\vec{s}(z)$ still lies on the Poincaré unit sphere after the transformation. The second equation in (27) is the Rodrigues formula for the rotation. This equation implies that the Stokes vector $\vec{s}(0)$ has been rotated around the vector \vec{p} by an angle $\delta(z)$ (Fig. 2). Without any loss of generality, we can always choose an \mathbb{R}^3 orthonormal basis $\{\vec{e}^i\}_{i\in\{1,2,3\}}$ such as $\vec{p}=\vec{e}^1$. Then the Stokes four-vector transformation (27) can be written in the following simple form:

$$\mathbf{s}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \delta(z) & \sin \delta(z) \\ 0 & 0 & -\sin \delta(z) & \cos \delta(z) \end{pmatrix} \mathbf{s}(0), \quad (28)$$

where we recognize the rotation matrix around the \vec{e}^1 axis by an angle $\delta(z)$.

The transformation (27) of the light wave polarization presented in Fig. 2 is typical of birefringent devices. For example, a $\lambda/4$ plate can be used to transform a linear polarized light wave into a circular polarized one—i.e., to transform a Stokes vector lying in the equatorial plane into a Stokes vector pointing toward a pole of the Poincaré sphere.

B. Absorbing polarizer: $\gamma \neq 0, \delta=0$

We consider now the nonunitary case $\gamma \neq 0, \delta=0, \forall z \in \mathbb{R}^+$. In such a case, the transformation (26) of the Stokes four-vector \mathbf{s} becomes

$$s^\mu(z) = \begin{cases} s^0(z) = e^{-\gamma(z)}[s^0(0)\cosh \gamma(z) + \sinh \gamma(z)\vec{s}(0) \cdot \vec{p}], \\ \vec{s}(z) = e^{-\gamma(z)}\{\vec{s}(0) + s^0(0)\vec{p} \sinh \gamma(z) + [\cosh \gamma(z) - 1][\vec{p} \cdot \vec{s}(0)]\vec{p}\}. \end{cases} \quad (29)$$

Naturally, the s^0 component giving the intensity of the light wave [first equation in (29)] is a decreasing function of the parameter γ . As $\gamma \rightarrow \infty$, the Stokes vector is brought along the direction of the \vec{p} vector since

$$\vec{s}(z) \propto \vec{p}, \quad \gamma \rightarrow +\infty. \quad (30)$$

For a finite value of γ , the Stokes vector is not completely brought along the direction of the vector \vec{p} ; the polarization of the light wave is then only partial. From now on, devices which transform the light wave polarization according to (29) will be designated as nonperfect polarizers (finite γ) or perfect polarizer ($\gamma \rightarrow \infty$).

1. Four-dimensional representation

Again, without any loss of generality, we can choose $\vec{p} = \vec{e}^1$. Then the transformation (29) reads

$$\mathbf{s}(z) = e^{-\gamma(z)} \begin{pmatrix} \cosh \gamma(z) & \sinh \gamma(z) & 0 & 0 \\ \sinh \gamma(z) & \cosh \gamma(z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{s}(0). \quad (31)$$

We recognize here a Lorentz boost in the $\vec{p} = \vec{e}^1$ direction; the

absorption term γ is then similar to the *rapidity* $\Phi = \arg \tanh v/c$ in special relativity. So, up to an attenuation factor $e^{-\gamma}$, the transformation (29) is a Lorentz boost in the direction of the polarizer vector \vec{p} . The Lorentz boost mixes only the intensity component $s^0(0)$ and the component along the polarizer vector—i.e., the projection $\vec{s}(0) \cdot \vec{p}$. The other components are left unchanged by the boost. Associated with the global attenuation $e^{-\gamma(z)}$, the boost attenuates the components of \vec{s} which are orthogonal to the polarizer vector \vec{p} : \vec{s} is progressively brought in the direction of the polarization vector \vec{p} (Fig. 3).

Note that this case exactly corresponds to using a polaroid film with $\gamma = (\alpha_e - \alpha_o) z$.

2. Poincaré sphere representation

The evolution of the light wave polarization can also be followed on the surface of the Poincaré sphere by studying the evolution of the normalized Stokes vector $\underline{\vec{s}}(z) = \vec{s}(z) / \|\vec{s}(z)\|$:

$$\underline{\vec{s}}^\mu(z) = \begin{cases} \underline{s}^0(z) = \underline{s}^0(0) = s^0(0) \equiv 1, \\ \underline{\vec{s}}(z) = \frac{\vec{s}(0) + \vec{p} \sinh \gamma(z) + [\cosh \gamma(z) - 1][\vec{p} \cdot \vec{s}(0)]\vec{p}}{s^0(0) \cosh \gamma(z) + \sinh \gamma(z) \vec{s}(0) \cdot \vec{p}}. \end{cases} \quad (32)$$

We remark easily that throughout the continuous transformation the Stokes vector $\vec{s}(z)$ is always orthogonal to the unit vector

$$\vec{q} = \frac{1}{s^0(0) \sin \alpha(0)} \vec{p} \times \vec{s}(0), \quad (33)$$

where $\alpha(z) = \angle(\vec{p}, \vec{s}(z))$. Indeed, for any parameter z , we can write

$$\vec{s}(0) \times \vec{s}(z) \propto \vec{q}. \quad (34)$$

The set of the Stokes vectors $\vec{s}(z)$ with $z \in \mathbb{R}^+$ defines a plane orthogonal to the vector \vec{q} and containing the center of the Poincaré sphere. This implies that the normalized Stokes vector $\underline{\vec{s}}(z)$ describes an arc of a great circle of the Poincaré sphere \mathcal{S}^2 (Fig. 4). The projection of the Stokes vector $\vec{s}(z)$ describes a geodesic of the \mathcal{S}^2 unit Poincaré sphere. Therefore, the normalized Stokes vector $\underline{\vec{s}}(z)$ is rotated around the axis \vec{q} by an angle $\delta(z) = \alpha(0) - \alpha(z)$ (see Fig. 4). Using the Rodrigues formula, we can rewrite the transformation (32) as a simple rotation

$$\underline{\vec{s}}^\mu(z) = \begin{cases} \underline{s}^0(z) = \underline{s}^0(0) \equiv 1, \\ \underline{\vec{s}}(z) = \underline{\vec{s}}(0) \cos \delta(z) + \underline{\vec{s}}(0) \times \vec{q} \sin \delta(z), \end{cases} \quad (35)$$

with

$$\sin \delta(z) = \sin \alpha(0) \cos \alpha(z) - \sin \alpha(z) \cos \alpha(0) \quad (36)$$

and

$$\cos \delta(z) = \cos \alpha(0) \cos \alpha(z) + \sin \alpha(z) \sin \alpha(0), \quad (37)$$

where

$$\cos \alpha(z) = \frac{\tanh \gamma(z) + \cos \alpha(0)}{1 + \tanh \gamma(z) \cos \alpha(0)} \quad (38)$$

and

$$\sin \alpha(z) = \frac{\sin \alpha(0)}{\cosh \gamma(z) + \sinh \gamma(z) \cos \alpha(0)}. \quad (39)$$

C. General polarizer: $\gamma \neq 0$, $\delta \neq 0$

Choosing again $\vec{p} = \vec{e}^1$, the transformation (26) for a general polarizer can be written in the following matrix form:

$$\mathbf{s}(z) = e^{-\gamma(z)} \begin{pmatrix} \cosh \gamma(z) & \sinh \gamma(z) & 0 & 0 \\ \sinh \gamma(z) & \cosh \gamma(z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \delta(z) & \sin \delta(z) \\ 0 & 0 & -\sin \delta(z) & \cos \delta(z) \end{pmatrix} \mathbf{s}(0). \quad (40)$$

Equation (40) shows clearly that the action of a polarizer on

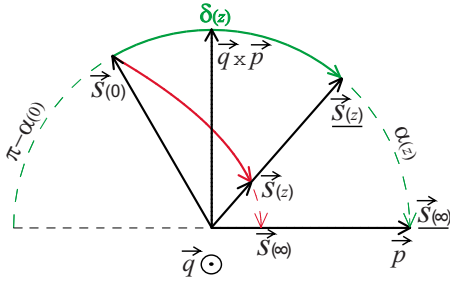


FIG. 4. (Color online) Representation of typical evolutions of the Stokes vector $\vec{s}(z)$ and of the normalized Stokes vector $\underline{s}(z)$ according to the transformations (29) and (32), respectively.

a light wave can be seen as three successive operations: a rotation of the Stokes vector $\vec{s}(0)$ by an angle $\delta(z)$ around the polarizer vector \vec{p} , a Lorentz boost in the direction of the polarizer vector \vec{p} , and the global attenuation $e^{-\gamma(z)}$. Each of these three operations evidently commutes with the others. Thus, the operator (22) associated with a general polarizing device can be rewritten as

$$P_{\vec{p}, \gamma, \delta}(z) = e^{-\gamma(z)/2} e^{-i\delta(z)/2} R_{\vec{q}, \delta}(z) B_{\vec{p}, \gamma}(z), \quad (41)$$

where in the $SL(2, \mathbb{C})$ group representation

$$B_{\vec{p}, \gamma}(z) = e^{\gamma(z)/2 \vec{p} \cdot \vec{\sigma}} \quad (42)$$

is the Lorentz boost operator in the direction \vec{p} with the rapidity γ and

$$R_{\vec{q}, \delta}(z) = e^{i\delta(z)/2 \vec{q} \cdot \vec{\sigma}} \quad (43)$$

is the rotation operator with the rotation vector \vec{q} and the angle of rotation δ . Figure 5 gives a typical illustration of the Stokes vector evolution for a general polarizer (26).

Note that this case corresponds to total reflection based polarizers (20) with $\gamma = \mu z$ and $\delta = -kz$.

As a conclusion of this section, we can state that during the polarization process of a wave light, the evolution of the Stokes vector—i.e., the evolution of the polarization—does not necessarily describe a geodesic on the Poincaré sphere. A geodesic—i.e., here a part of a great circle of \mathcal{S}^2 —is found only in the case of polarizing device such as a $\lambda/4$ plate or in the case of a nonunitary polarizer with $\delta=0$ such as, e.g., a Polaroid film. In this last case, it is the projection of the Stokes vector on the Poincaré sphere which describe a geodesic. All these statements on the effective trajectory of the light wave polarization are worth considering when geometric phase has to be calculated [11].

IV. GENERALIZED MALUS LAW, DEGREE OF POLARIZATION

Now, let us look at the intensity component $s^0(z)$ of the Stokes four-vector. We define $\alpha(z) = 2\theta(z)$ as the oriented angle $\angle(\vec{p}, \vec{s}(z))$ between the polarizer vector \vec{p} and the Stokes vector $\vec{s}(z)$ for the parameter z . Then, $\theta(z)$ is the angle, in the laboratory frame, between the axis of the polarizer and the ordinary axis of the light wave at position z . The intensity of the light wave, given by the first equation in (26), can be rewritten as

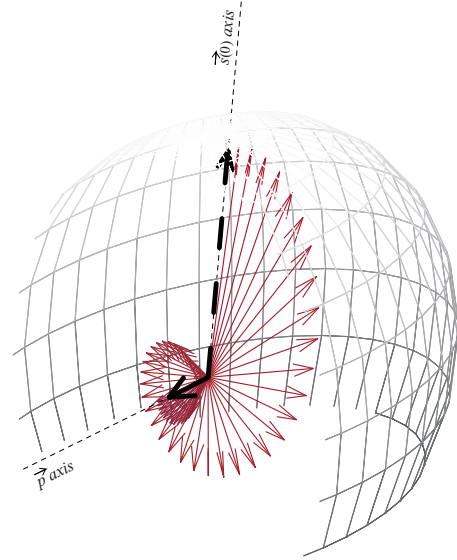


FIG. 5. (Color online) Typical representation of the Stokes vector $\vec{s}(z)$ evolution according to the general transformation (40) or (26). The two dashed lines give the direction of the initial Stokes vector $\vec{s}(0)$ and the direction of the polarization vector \vec{p} , respectively. The initial Stokes vector $\vec{s}(0)$ is represented by the thick black dashed vector. The Stokes vector for $\gamma \rightarrow \infty$ is represented by the thick black vector oriented along the direction of the polarization vector \vec{p} . The other vectors are the successive Stokes vectors \vec{s} as the parameter γ (or equivalently the parameter z) increases from 0 to ∞ .

$$I(z) = I(0)[\cos^2 \theta(0) + \sin^2 \theta(0)e^{-2\gamma(z)}], \quad (44)$$

which is a generalization of the Malus law for nonperfect polarizers. Figure 6(a) shows the Malus law for different value of the absorption term $\gamma(z)$ from 0 to $+\infty$. Let us define $I_{\parallel}(z) = I_{\vec{p}}(z)$ as the intensity of the light wave in the $\rho_{\vec{p}}$ polarizer state and $I_{\perp}(z) = I_{-\vec{p}}(z)$ as the intensity in the corresponding orthogonal state $\rho_{-\vec{p}}$. After some calculus we obtain

$$I_{\parallel}(z) = I_{\vec{p}}(z) = \text{Tr}(\rho_{\vec{s}(z)} \rho_{\vec{p}}) = I(0) \cos^2 \theta(0) \quad (45)$$

and

$$I_{\perp}(z) = I_{-\vec{p}}(z) = \text{Tr}(\rho_{\vec{s}(z)} \rho_{-\vec{p}}) = I(0) \sin^2 \theta(0) e^{-2\gamma(z)}. \quad (46)$$

Now, if the polarizer is placed in an unpolarized beam, the transmitted intensity at the point z is

$$\langle I(z) \rangle_{\theta(0)} = \langle I_{\parallel}(z) \rangle_{\theta(0)} + \langle I_{\perp}(z) \rangle_{\theta(0)} = \frac{I(0)}{2} (1 + e^{-2\gamma(z)}), \quad (47)$$

where the brackets $\langle \dots \rangle_{\theta(0)}$ denote the average over the initial angles $\theta(0)$. From (47), we remark that for a nonperfect polarizer (γ finite) more than half of the incoming intensity is transmitted. The degree of polarization $\eta(z)$ of the nonperfect polarizer is given by

$$\eta(z) = \frac{\langle I_{\parallel}(z) \rangle_{\theta(0)} - \langle I_{\perp}(z) \rangle_{\theta(0)}}{\langle I_{\parallel}(z) \rangle_{\theta(0)} + \langle I_{\perp}(z) \rangle_{\theta(0)}} = \tanh \gamma(z) \quad (48)$$

and the extinction ratio $\xi(z)$ by

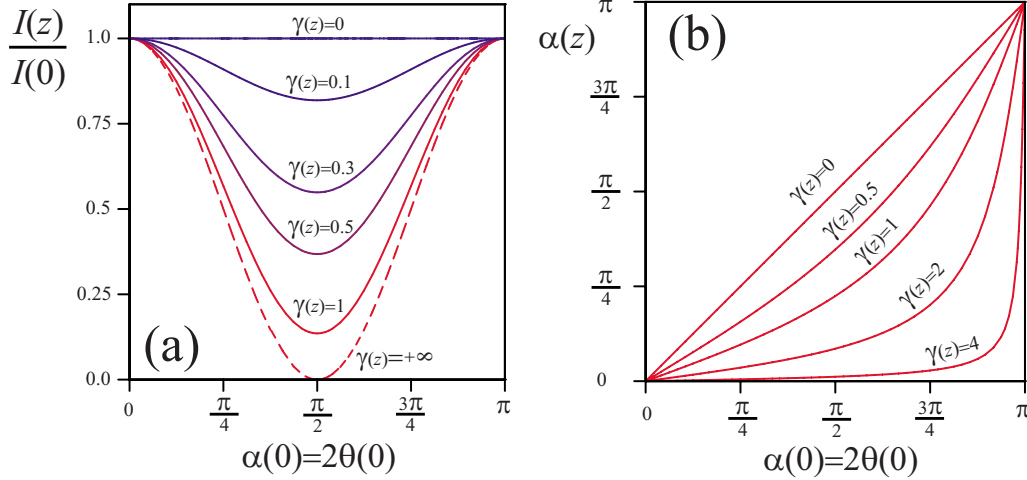


FIG. 6. (Color online) (a) Generalized Malus law: representation of the ratio $I(z)/I(0)$ as a function of $\alpha(0)=2\theta(0)$ [angle between the polarizer vector \vec{p} and the incoming wave polarization $\vec{s}(0)$] for different values of the absorption term $\gamma(z)$. (b) Angle $\alpha(z)$ between the polarizer vector \vec{p} and the wave polarization $\vec{s}(z)$ as a function of $\alpha(0)$ for different values of the absorption term $\gamma(z)$.

$$\xi(z) = \frac{\langle I_{\perp}(z) \rangle_{\theta(0)}}{\langle I_{\parallel}(z) \rangle_{\theta(0)}} = e^{-2\gamma(z)}. \quad (49)$$

As $\gamma(z)$ varies from 0 to $+\infty$, $\eta(z)$ and $\xi(z)$ vary from 0 to 1 and from 1 to 0, respectively. Although $\eta(z)$ and $\xi(z)$ measure already the efficiency of the polarizer, we can always ask *how far* is the polarization vector $\vec{s}(z)$ from the polarization vector \vec{p} associated with the polarizer. For that purpose we focus on the angle $\alpha(z)$ between the two vectors. Combining the two equations in (26), or equivalently using (48) and (38), we obtain the relation

$$\cos \alpha(z) = \frac{\eta(z) + \cos \alpha(0)}{1 + \eta(z) \cos \alpha(0)}, \quad (50)$$

from which we are able to extract the angle $\alpha(z)$ as

$$\alpha(z) = \arccos(\tanh\{\gamma(z) + \arg \tanh[\cos \alpha(0)]\}). \quad (51)$$

Thus, as $\gamma(z)$ varies from 0 to $+\infty$, the angle $\alpha(z)$ varies from $\alpha(0)$ to 0, or equivalently $\cos \alpha(z)$ varies from $\cos \alpha(0)$ to 1. Figure 6(b) shows $\alpha(z)$ as a function of $\alpha(0)$ for different values of the absorption term $\gamma(z)$. Expressions (50) and (51) are dependent on the initial angle $\alpha(0)$. The integration of (50) over all the possible initial angles $\alpha(0)$, i.e.,

$$\langle \cos \alpha(z) \rangle_{\alpha(0)} = \frac{1}{\pi} \int_0^{\pi} d\alpha(0) \cos \alpha(z), \quad (52)$$

provides a convenient parameter to characterize the quality of the polarizer. Analytic calculation of (52) gives us

$$\langle \cos \alpha(z) \rangle_{\alpha(0)} = \frac{1 - \sqrt{1 - \eta^2(z)}}{\eta(z)} = \tanh \frac{\gamma(z)}{2}. \quad (53)$$

Hence, as $\gamma(z)$ varies from 0 to $+\infty$ (perfect polarizer), the parameter $\langle \cos \alpha(z) \rangle_{\alpha(0)}$ varies from 0 to 1.

V. COMPOSITION LAW FOR POLARIZERS

From now on, we assume the z parameter dependence of the different functions, such as, e.g., δ or γ , and for clarity

we drop the z parameter from the following mathematical expressions.

A. Association of two polarizers

Let us now consider the successive actions of two absorbing polarizers ($\delta=0$; see Sec. III B) with different polarization axes \vec{p}_1 and \vec{p}_2 and different absorption terms γ_1 and γ_2 . The corresponding operator of the total system is then

$$P_{\vec{p}_2, \gamma_2, 0} P_{\vec{p}_1, \gamma_1, 0} = e^{-(\gamma_1 + \gamma_2)/2} B_{\vec{p}_2, \gamma_2} B_{\vec{p}_1, \gamma_1}, \quad (54)$$

where we use the $SL(2, \mathbb{C})$ group representation (42) of the Lorentz boost operator. We keep in mind that the case of the composition of two general polarizers—i.e., with $\delta \neq 0$ (see Sec. III C)—can be easily retrieved from the case (54) of two absorbing polarizers ($\delta=0$). We focus now on the two successive boosts in (54). We know (see, e.g., [6,7]) that the action of two successive noncollinear Lorentz boosts $B_{\vec{p}_1, \gamma_1}$ and $B_{\vec{p}_2, \gamma_2}$ is equivalent to the action of a Lorentz boost $B_{\vec{p}, \gamma}$ followed by a rotation $R_{\vec{q}, \delta}$; i.e.,

$$B_{\vec{p}_2, \gamma_2} B_{\vec{p}_1, \gamma_1} = R_{\vec{q}, \delta} B_{\vec{p}, \gamma}, \quad (55)$$

with suitable parameters γ and δ and suitable directions \vec{p} and \vec{q} . Then, (54) can be rewritten as

$$P_{\vec{p}_2, \gamma_2, 0} P_{\vec{p}_1, \gamma_1, 0} = e^{-(\gamma_1 + \gamma_2)/2} R_{\vec{q}, \delta} B_{\vec{p}, \gamma}, \quad (56)$$

or, using only polarizing device operators (22), as

$$P_{\vec{p}_2, \gamma_2, 0} P_{\vec{p}_1, \gamma_1, 0} = e^{(\gamma_1 - \gamma_2)/2} e^{i\delta/2} P_{\vec{q}, \delta, 0} P_{\vec{p}, \gamma, 0}. \quad (57)$$

In the Appendix, we derive from the initial parameters γ_1 , γ_2 , \vec{p}_1 , and \vec{p}_2 the expressions of the resulting rotation parameters δ and \vec{q} and of the resulting boost parameters γ and \vec{p} entering the relation (55) and consequently entering (56) and (57). For the rotation parameters we have (A6),

$$\vec{q} = \frac{1}{\sin \alpha} \vec{p}_2 \times \vec{p}_1, \quad (58)$$

and (A9),

$$\prod_{j=1}^{N-1} \widehat{\mathcal{R}}(j, \dots, N) = R_{\vec{q}, \delta} = e^{i\delta/2\vec{q}\cdot\vec{\sigma}} \quad (68)$$

is a resulting rotation operator characterized by a rotation vector \vec{q} and an angle δ calculated from successive applications of (A9). The operator (64) corresponding to N successive absorbing polarizers can then be rewritten as

$$\mathcal{P}(1, 2, \dots, N) = \exp\left(-\frac{1}{2}\sum_{j=1}^N \gamma_j\right) R_{\vec{q}, \delta} \mathcal{B}_{\vec{p}, \gamma} \quad (69)$$

or using polarizing device operators (22) as

$$\mathcal{P}(1, 2, \dots, N) = \exp\left[\frac{1}{2}\left(\gamma - \sum_{j=1}^N \gamma_j\right)\right] e^{i\delta/2} P_{\vec{q}, 0, \delta} \mathcal{P}_{\vec{p}, \gamma, 0}. \quad (70)$$

Using (69), the initial state $\rho_{\vec{s}_0}$ of an incoming light wave is transformed into the following final state:

$$\rho_{\vec{s}} = \mathcal{P}(1, \dots, N) \rho_{\vec{s}_0} \mathcal{P}^\dagger(1, \dots, N), \quad (71)$$

with the corresponding intensity

$$I = \text{Tr}(\rho_{\vec{s}}) = I_0 \exp\left(\gamma - \sum_{j=1}^N \gamma_j\right) (\cos^2 \theta_0 + \sin^2 \theta_0 e^{-2\gamma}). \quad (72)$$

Here $2\theta_0$ (see Sec. IV) is the angle between the polarization \vec{s}_0 of the incoming wave and the resulting polarization vector \vec{p} . So the transmitted intensity of an unpolarized light beam is

$$\langle I \rangle_{\theta_0} = \frac{I_0}{2} \exp\left(\gamma - \sum_{j=1}^N \gamma_j\right) (1 + e^{-2\gamma}). \quad (73)$$

Let us define $\eta_{\vec{p}}$ as the degree of polarization of the N polarizers using the resulting polarization state $\rho_{\vec{p}}$ as the state of reference. Using

$$I_{\vec{p}} = \text{Tr}(\rho_{\vec{s}} \rho_{\vec{p}}) = I_0 \exp\left(\gamma - \sum_{j=1}^N \gamma_j\right) \cos^2 \theta_0 \quad (74)$$

and

$$I_{-\vec{p}} = \text{Tr}(\rho_{\vec{s}} \rho_{-\vec{p}}) = I_0 \exp\left(-\gamma - \sum_{j=1}^N \gamma_j\right) \sin^2 \theta_0, \quad (75)$$

we obtain the corresponding degree of polarization

$$\eta_{\vec{p}} = \frac{\langle I_{\vec{p}} \rangle_{\theta_0} - \langle I_{-\vec{p}} \rangle_{\theta_0}}{\langle I_{\vec{p}} \rangle_{\theta_0} + \langle I_{-\vec{p}} \rangle_{\theta_0}} = \tanh \gamma \quad (76)$$

and the corresponding extinction ratio

$$\xi_{\vec{p}} = \frac{\langle I_{-\vec{p}} \rangle_{\theta_0}}{\langle I_{\vec{p}} \rangle_{\theta_0}} = e^{-2\gamma}. \quad (77)$$

The expressions found in (76) and (77) are similar to the expressions of the degree of polarization η , (48), and of the extinction ratio ξ , (49), found for one polarizer except that

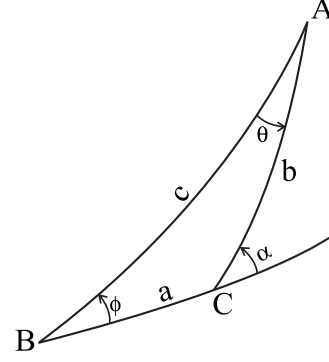


FIG. 8. The hyperbolic triangle ABC corresponding to the polarizers P_1 , P_2 and to the resulting polarizer P in the Poincaré disk.

here the absorption term γ results from the action of N absorbing polarizers. So the composition law (62), whose $N-1$ successive applications give $\Gamma = \tanh \frac{\gamma}{2}$, is also the composition law for the degree of polarization η , (76).

To conclude this section let us retrieve the intensity of an initially unpolarized light beam through the succession of two perfect polarizers ($\gamma_1 \rightarrow \infty$, $\gamma_2 \rightarrow \infty$). First, the modulus of the composition law (60) implies that $\gamma \rightarrow \infty$. In that case the expression (73) becomes

$$\langle I \rangle_{\theta_0} = \frac{I_0}{2} e^{\gamma - \gamma_1 - \gamma_2}. \quad (78)$$

Using (A2) and (A8) in the case of two perfect polarizers ($\gamma_1 \rightarrow \infty$, $\gamma_2 \rightarrow \infty$, $\gamma \rightarrow \infty$) we obtain the following relation

$$\cos^2 \frac{\alpha}{2} = e^{\gamma - \gamma_1 - \gamma_2}, \quad (79)$$

where $\alpha = 2\theta_{12}$ is the angle between the polarization vectors \vec{p}_1 and \vec{p}_2 (in the Poincaré sphere representation) of the two successive polarizers. Thus (78) can be rewritten as

$$\langle I \rangle_{\theta_0} = \frac{I_0}{2} \cos^2 \theta_{12}, \quad (80)$$

which is the well-known expression for the intensity of an initially unpolarized light beam going through two perfect polarizers. We remark that γ given by the composition law (61) encodes in (78) and more generally in (73) the relative positions on the Poincaré sphere of the successive polarizer vectors \vec{p}_j .

C. Discussion

The result (60) can be illustrated on the upper sheet H^+ of an hyperboloid or, more easily, on its stereographic projection known as the Poincaré disk. Let us consider in Fig. 8 the hyperbolic triangle ABC , the three sides a , b , and c of which correspond to polarizers P_1 , P_2 and to the resulting polarizer P , respectively. In that triangle, the lengths of sides a , b , and c , opposite to vertices A , B , and C , are $\tanh \frac{\gamma_1}{2}$, $\tanh \frac{\gamma_2}{2}$, and $\tanh \frac{\gamma}{2}$, respectively. They physically correspond to the *quality* of the corresponding polarizers: the length of each side approaches the limit 1 when its corresponding γ tends to ∞ .

In that triangle (see Fig. 8), α is the angle between the axis of P_1 and the axis of P_2 , ϕ is the angle between the axis of P_1 and the axis of the resulting polarizer P , and θ is the angle between the axis of P and the axis of P_2 . For the sake of simplicity, we choose the axis of reference along the direction of the first polarizers—i.e., $\vec{u}=\vec{p}_1$ —so that in Fig. 7 we have $\alpha_1=0$ and $\alpha_2=\alpha$.

Now, in order to understand (60), let us use Fig. 8 when the second polarizer P_2 is perfect; i.e., $b=\tanh \frac{\gamma_2}{2}=1$. In such a case, (60) reduces to

$$\tanh \frac{\gamma}{2} e^{i\phi} = \frac{\tanh \frac{\gamma_1}{2} + e^{i\alpha}}{1 + \tanh \frac{\gamma_1}{2} e^{i\alpha}}. \quad (81)$$

So, using (59), we obtain

$$\tanh \frac{\gamma}{2} e^{i\phi} = e^{i\alpha} e^{-i\delta}. \quad (82)$$

This result shows that the polarizer equivalent to using polarizers P_1 and P_2 successively (and in that order) is a perfect polarizer ($\tanh \frac{\gamma}{2}=1$) the axis of which is in the direction $\phi = \alpha - \delta$. That result could be surprising, since we could expect to find its axis in the α direction. This comes from properties of hyperbolic triangles. The angle θ between P and P_2 can be calculated using (60): noting that the angle of AC with CB is $-\alpha$ and taking now the direction of the arbitrary reference axis for angles in the direction of P_2 , we get (note that $\tanh \frac{\gamma_2}{2}=1$)

$$\tanh \frac{\gamma}{2} e^{-i\theta} = \frac{\tanh \frac{\gamma_2}{2} + \tanh \frac{\gamma_1}{2} e^{-i\alpha}}{1 + \tanh \frac{\gamma_2}{2} \tanh \frac{\gamma_1}{2} e^{-i\alpha}} = \frac{1 + \tanh \frac{\gamma_1}{2} e^{-i\alpha}}{1 + \tanh \frac{\gamma_1}{2} e^{-i\alpha}} = 1, \quad (83)$$

so that $\theta=0$: the axis of P equivalent to using P_1 and P_2 successively is, as expected, directed in the direction of the axis of P_2 . These two last results—i.e., the angle of P with P_1 is $\phi=\alpha-\delta$ and that of P with P_2 is $\theta=0$ —come from the fact that the sum of the three angles of an hyperbolic triangle is not π , but $\pi-\delta$ where δ is its angular defect. It is interesting to underline that this case (P_2 perfect) corresponds to the case of aberration in special relativity.

For the sake of completeness, let us show that δ is indeed the angular defect, which we denote δ' , of the hyperbolic triangle ABC . The sum of the angles of the hyperbolic triangle ABC gives

$$\phi + (\pi - \alpha) + \theta = \pi - \delta', \quad (84)$$

so that

$$e^{i\delta'} = e^{i\alpha} e^{-i\phi} e^{-i\theta}. \quad (85)$$

Using (A16) and extracting the angle θ from the first equality of (83), (85) can be rewritten as

$$e^{i\delta'} = \frac{1 + \tanh \frac{\gamma_1}{2} \tanh \frac{\gamma_1}{2} e^{i\alpha}}{1 + \tanh \frac{\gamma_1}{2} \tanh \frac{\gamma_1}{2} e^{-i\alpha}}. \quad (86)$$

Comparing (86) with (59) gives us $\delta' = \delta$. Thus the parameter δ of the rotation $R_{\vec{q},\delta}$ (55) is the angular defect of the hyperbolic triangle build using the parameters of P_1 , P_2 , and P .

To conclude this section, let us remark that the angle ϕ , (A16), giving the direction of the resulting polarizer P can be rewritten as

$$\phi = \phi_{\text{Euclid}} - \frac{\delta}{2}, \quad (87)$$

where

$$\phi_{\text{Euclid}} = \arg \left(\tanh \frac{\gamma_1}{2} + \tanh \frac{\gamma_2}{2} e^{i\alpha} \right) \quad (88)$$

would have been the angle between the axis of the polarizers P_1 and P if the geometry were Euclidean.

VI. WIGNER ANGLE

The composition law (62)—i.e., the addition law \oplus —is noncommutative [13]. Indeed, the final polarization of the light wave passing through polarizer 1 and then through polarizer 2 is not the same as the final polarization of the light wave passing through polarizer 2 and then through polarizer 1. Such a noncommutativity is obvious in the case of perfect polarizers since the final polarization always corresponds to the polarization vector of the second polarizer. With two polarizers with respective absorption terms γ_1 and γ_2 and with respective vectors \vec{p}_1 and \vec{p}_2 , the two possibilities of composition are

$$\Gamma_{1\oplus 2} e^{i\phi_{1\oplus 2}} = \Gamma_1 e^{i\alpha_1} \oplus \Gamma_2 e^{i\alpha_2} \quad (89)$$

and

$$\Gamma_{2\oplus 1} e^{i\phi_{2\oplus 1}} = \Gamma_2 e^{i\alpha_2} \oplus \Gamma_1 e^{i\alpha_1}. \quad (90)$$

From the definition of the addition law (61) we easily see that

$$\Gamma_{1\oplus 2} = \Gamma_{2\oplus 1} = \Gamma \quad (91)$$

and consequently

$$\gamma_{1\oplus 2} = \gamma_{2\oplus 1} = \gamma. \quad (92)$$

However, the angles $\phi_{1\oplus 2}$ and $\phi_{2\oplus 1}$ are not equal. Thus, by analogy with special relativity (or with optical studies of multilayers [12]), we introduce the Wigner angle $w = \phi_{2\oplus 1} - \phi_{1\oplus 2}$ which can be seen as a measure of the noncommutativity of the addition law \oplus . Using (61), (89), and (90), we easily determine the Wigner angle [14] (see also [15,16]) through

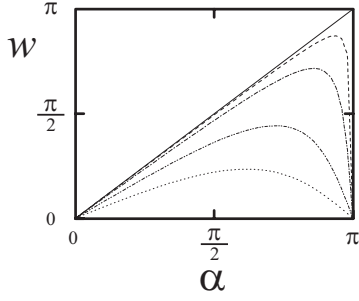


FIG. 9. Representation of the Wigner angle w , (94), as a function of α for different values of the product $\sqrt{\Gamma_1 \Gamma_2}$ from 0.6 (dotted line), 0.8 (double-dot-dashed line), 0.95 (dot-dashed line), 0.99 (dashed line), and 1 (solid line).

$$e^{iw} = \frac{\Gamma_2 e^{i\alpha_2} \oplus \Gamma_1 e^{i\alpha_1}}{\Gamma_1 e^{i\alpha_1} \oplus \Gamma_2 e^{i\alpha_2}} = \frac{1 + \Gamma_1 \Gamma_2 e^{i\alpha}}{1 + \Gamma_1 \Gamma_2 e^{-i\alpha}} = \Gamma_1 \Gamma_2 e^{i\alpha} \oplus 1, \quad (93)$$

which gives then

$$w = 2 \arg(1 + \Gamma_1 \Gamma_2 e^{i\alpha}) = \delta, \quad (94)$$

where $\alpha = \alpha_2 - \alpha_1$ is the angle between \vec{p}_1 and \vec{p}_2 .

For perfect polarizers $\Gamma_1 \Gamma_2 = 1$, the light wave is polarized first in one direction \vec{p}_1 and then in the other direction \vec{p}_2 , so the Wigner angle is $w = \alpha$. For nonperfect polarizers the measure of the Wigner angle w can be seen as a measure of the quality of the polarizers. Indeed, if one measures $w < \alpha$ for a given α , we are able to retrieve the product $\Gamma_1 \Gamma_2$ and then to state on the quality of the polarizers. Figure 9 shows the Wigner angle w as a function of α for different values of the product $\Gamma_1 \Gamma_2$.

VII. CONCLUSION

Textbooks usually only consider perfect polarizers (i.e., polarizers with $\gamma \rightarrow \infty$) and do not examine the question of knowing how the polarization of a light beam is progressively transformed when propagating inside a polarizer. In the present paper, we have addressed these two problems using the notion of the Stokes vector and that of the Poincaré sphere in order to characterize the evolution of a light wave polarization through a polarizer. Whereas the action of a polarizer is intuitively thought of to be a simple rotation of the polarization along a geodesic on the Poincaré sphere, we have shown that it is not always the case. In the general case, more complex trajectories differing from geodesics can in fact be found, especially in the case of total-reflection-based polarizers (see Sec. III C). How these trajectories will affect geometric phases of light wave going through polarizers will be a question addressed in a following paper [11].

Nonperfect polarizers—i.e., here polarizers with finite γ —lead us naturally to define a generalized Malus law and a degree of polarization $\eta(z)$.

The action of a polarizer on light wave polarization can be described by Lorentz operators, where the attenuation factor γ is the counterpart of the rapidity Φ commonly used in special relativity. The association of two such polarizers, in-

volving then two Lorentz boosts, can always be viewed as the result of a Lorentz boost transformation followed by a rotation. In other words, the association of two polarizers is equivalent to the association of a polarizer and, for example, a device with optical activity giving an additional rotation of the polarization. We have then derived a composition law (61) for polarizers similar to the composition law for noncollinear velocities in special relativity. This composition law \oplus appears to be a very natural law to “add” quantities the values of which are limited to finite values. In fact, (61) implies that no matter what are the values of the complex quantities a and b , such as $|a| < 1$ and $|b| < 1$, the modulus of the overall resulting quantity $a \oplus b$ cannot exceed unity. Because it avoids infinity, such a generalization of Einstein’s composition law of velocities appears to be a natural “addition” law of physical quantities in a closed interval.

The association of N polarizers can also be viewed as the association of a resulting polarizer and an optical device giving an additional rotation. Recursive iterations of the composition law (61) give directly the characteristics of the resulting polarizer (i.e., γ and \vec{p}) and of its associate rotation (i.e., δ and \vec{q}). We have defined the Malus law for the association of N polarizers and the corresponding degree of polarization, the latter being easily determined using the composition law.

In addition to the degree of polarization (or equivalently the extinction ratio) we have defined others quantities characterizing the quality of the polarizers. We have defined (see Sec. IV) the angle $\alpha(z)$ measuring how far the light wave polarization vector is from the polarizer vector \vec{p} . Also, using the existing isomorphism between special relativity and polarization in optics, we have defined (see Sec. VI) the relativistic Wigner angle for the association of two polarizers. The Wigner angle is a direct consequence of the noncommutativity of the addition law \oplus , (61), and we have shown that the latter can be used to directly measure the quality of the association of two polarizers.

APPENDIX

We aim to determine the absorption term γ , the angle δ , and the vectors \vec{p} and \vec{q} entering the relation (55); i.e.,

$$B_{\vec{p}_2, \gamma_2} B_{\vec{p}_1, \gamma_1} = R_{\vec{q}, \delta} B_{\vec{p}, \gamma}. \quad (A1)$$

We define $\alpha = \angle(\vec{p}_1, \vec{p}_2) \in [0, \pi]$ as the angle between the two polarizer vectors \vec{p}_1 and \vec{p}_2 . Using the definitions (42) and (43) and expanding the exponential operators in (A1) we obtain the following set of four equations:

$$\cosh \frac{\gamma_2}{2} \cosh \frac{\gamma_1}{2} + \sinh \frac{\gamma_2}{2} \sinh \frac{\gamma_1}{2} \cos \alpha = \cos \frac{\delta}{2} \cosh \frac{\gamma}{2}, \quad (A2)$$

$$\sin \frac{\delta}{2} \sinh \frac{\gamma}{2} \vec{q} \cdot \vec{p} = 0, \quad (A3)$$

$$\begin{aligned} & \sinh \frac{\gamma_2}{2} \cosh \frac{\gamma_1}{2} \vec{p}_2 + \sinh \frac{\gamma_1}{2} \cosh \frac{\gamma_2}{2} \vec{p}_1 \\ &= \sinh \frac{\gamma}{2} \cos \frac{\delta}{2} \vec{p} - \sin \frac{\delta}{2} \sinh \frac{\gamma}{2} \vec{q} \times \vec{p}, \end{aligned} \quad (\text{A4})$$

$$\sinh \frac{\gamma_2}{2} \sinh \frac{\gamma_1}{2} \vec{p}_2 \times \vec{p}_1 = \sin \frac{\delta}{2} \cosh \frac{\gamma}{2} \vec{q}. \quad (\text{A5})$$

1. Rotation parameters δ, \vec{q}

As (A3) holds for any parameter $\gamma \in \mathbb{R}^+$ and for any angle $\delta \in [0, 2\pi[$, we have $\vec{q} \cdot \vec{p} = 0$. Consequently, the two vectors \vec{p} and \vec{q} are orthogonal. Equation (A5) fixes then the unit vector \vec{q} as

$$\vec{q} = \frac{1}{\sin \alpha} \vec{p}_2 \times \vec{p}_1. \quad (\text{A6})$$

The vectors \vec{p}_1, \vec{p}_2 , and \vec{p} are coplanar since \vec{p} is orthogonal to \vec{q} (see Fig. 7). Using (A6), (A5) can be replaced by the following equation:

$$\sinh \frac{\gamma_2}{2} \sinh \frac{\gamma_1}{2} \sin \alpha = \sin \frac{\delta}{2} \cosh \frac{\gamma}{2}. \quad (\text{A7})$$

With (A2) and (A7) the angle δ is given by

$$\tan \frac{\delta}{2} = \frac{\tanh \frac{\gamma_1}{2} \tanh \frac{\gamma_2}{2} \sin \alpha}{1 + \tanh \frac{\gamma_1}{2} \tanh \frac{\gamma_2}{2} \cos \alpha} \quad (\text{A8})$$

or, equivalently, by

$$\frac{\delta}{2} = \arg \left(1 + \tanh \frac{\gamma_1}{2} \tanh \frac{\gamma_2}{2} e^{i\alpha} \right). \quad (\text{A9})$$

Thus, from the initial parameters, $\gamma_1, \gamma_2, \vec{p}_1$, and \vec{p}_2 , (A6) and (A9) can be used to obtain respectively the axis \vec{q} and the angle δ of the rotation entering (A1).

2. Boost parameters γ, \vec{p}

Let us now define the angles $\alpha_1 = \angle(\vec{u}, \vec{p}_1)$ and $\alpha_2 = \angle(\vec{u}, \vec{p}_2)$ (see Fig. 7) where \vec{u} is an arbitrary reference axis in the plane defined by \vec{p}_1 and \vec{p}_2 . In order to calculate the angle $\phi = \angle(\vec{u}, \vec{p})$, we project the two members of (A4) along

the direction \vec{u} and along the orthogonal direction $\vec{u} \times \vec{q}$. We obtain the following two equations:

$$\begin{aligned} & \sinh \frac{\gamma_2}{2} \cosh \frac{\gamma_1}{2} \cos \alpha_2 + \sinh \frac{\gamma_1}{2} \cosh \frac{\gamma_2}{2} \cos \alpha_1 \\ &= \cos \frac{\delta}{2} \sinh \frac{\gamma}{2} \cos \phi - \sin \frac{\delta}{2} \sinh \frac{\gamma}{2} \sin \phi, \end{aligned} \quad (\text{A10})$$

$$\begin{aligned} & \sinh \frac{\gamma_2}{2} \cosh \frac{\gamma_1}{2} \sin \alpha_2 + \sinh \frac{\gamma_1}{2} \cosh \frac{\gamma_2}{2} \sin \alpha_1 \\ &= \cos \frac{\delta}{2} \sinh \frac{\gamma}{2} \sin \phi + \sin \frac{\delta}{2} \sinh \frac{\gamma}{2} \cos \phi. \end{aligned} \quad (\text{A11})$$

The combination of these two equations gives us

$$\sinh \frac{\gamma_2}{2} \cosh \frac{\gamma_1}{2} e^{i\alpha_2} + \sinh \frac{\gamma_1}{2} \cosh \frac{\gamma_2}{2} e^{i\alpha_1} = \sinh \frac{\gamma}{2} e^{i(\phi + \delta/2)}. \quad (\text{A12})$$

Combining now (A2) and (A7), we obtain

$$\cosh \frac{\gamma_2}{2} \cosh \frac{\gamma_1}{2} + \sinh \frac{\gamma_2}{2} \sinh \frac{\gamma_1}{2} e^{i\alpha} = \cosh \frac{\gamma}{2} e^{i\delta/2}. \quad (\text{A13})$$

Dividing the members of (A12) by those of (A13), we find

$$\tanh \frac{\gamma}{2} e^{i\phi} = \frac{\tanh \frac{\gamma_1}{2} e^{i\alpha_1} + \tanh \frac{\gamma_2}{2} e^{i\alpha_2}}{1 + \tanh \frac{\gamma_1}{2} \tanh \frac{\gamma_2}{2} e^{i(\alpha_2 - \alpha_1)}}. \quad (\text{A14})$$

From the initial parameters $\gamma_1, \gamma_2, \vec{p}_1$, and \vec{p}_2 , (A14) allows us to calculate the absorption term γ and the direction \vec{p} entering (A1). Indeed, the modulus of (A14) gives us γ since

$$\tanh \frac{\gamma}{2} = \left| \frac{\tanh \frac{\gamma_1}{2} + \tanh \frac{\gamma_2}{2} e^{i(\alpha_2 - \alpha_1)}}{1 + \tanh \frac{\gamma_1}{2} \tanh \frac{\gamma_2}{2} e^{i(\alpha_2 - \alpha_1)}} \right| \quad (\text{A15})$$

and the direction of the unit vector \vec{p} is determined from

$$\phi = \alpha_1 + \arg \left(\frac{\tanh \frac{\gamma_1}{2} + \tanh \frac{\gamma_2}{2} e^{i(\alpha_2 - \alpha_1)}}{1 + \tanh \frac{\gamma_1}{2} \tanh \frac{\gamma_2}{2} e^{i(\alpha_2 - \alpha_1)}} \right). \quad (\text{A16})$$

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