

From Feynman proof of Maxwell equations to noncommutative quantum mechanics

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Abstract. In 1990, Dyson published a proof due to Feynman of the Maxwell equations assuming only the commutation relations between position and velocity. With this minimal assumption, Feynman never supposed the existence of Hamiltonian or Lagrangian formalism. In the present communication, we review the study of a relativistic particle using “Feynman brackets.” We show that Poincaré’s magnetic angular momentum and Dirac magnetic monopole are the consequences of the structure of the Lorentz Lie algebra defined by the Feynman’s brackets. Then, we extend these ideas to the dual momentum space by considering noncommutative quantum mechanics. In this context, we show that the noncommutativity of the coordinates is responsible for a new effect called the spin Hall effect. We also show its relation with the Berry phase notion. As a practical application, we found an unusual spin-orbit contribution of a nonrelativistic particle that could be experimentally tested. Another practical application is the Berry phase effect on the propagation of light in inhomogeneous media.

1. Introduction

Various ways exist to present the Maxwell equations. The usual one is the historical approach in which the empirical basis for each equation is initially given. Another remarkable way is exposed in an old unpublished work of Feynman, reported in an elegant paper by Dyson [1] published in 1990. The initial Feynman’s motivation was to develop a quantization procedure without resorting to a Lagrangian or a Hamiltonian. For this, let us consider a nonrelativistic particle of mass m subjected to an external force $m \frac{d\dot{x}^i}{dt} = F^i(\mathbf{x}, \dot{\mathbf{x}}, t)$ and the fiber tangent space with a symplectic structure defined by the “Feynman brackets” $[x^i, x^j] = 0$ and $m [x^i, \dot{x}^j] = \delta^{ij}$. The Feynman brackets obey the Leibnitz law

$$\frac{d}{dt} [f(\mathbf{x}, \dot{\mathbf{x}}), g(\mathbf{x}, \dot{\mathbf{x}})] = \left[\frac{df(\mathbf{x}, \dot{\mathbf{x}})}{dt}, g(\mathbf{x}, \dot{\mathbf{x}}) \right] + \left[f(\mathbf{x}, \dot{\mathbf{x}}), \frac{dg(\mathbf{x}, \dot{\mathbf{x}})}{dt} \right] \quad (1)$$

and the Jacobi identity

$$\left[x^i, \left[x^j, x^k \right] \right] + \left[x^j, \left[x^k, x^i \right] \right] + \left[x^k, \left[x^i, x^j \right] \right] = 0. \quad (2)$$

From these assumptions, Feynman, in 1948, deduced the following relations :

$$\left[\dot{x}^i, \dot{x}^j \right] = F^{ij}(\mathbf{x}, t) = \varepsilon^{ijk} B_k(\mathbf{x}, t) \quad (3)$$

$$F^i(\mathbf{x}, \dot{\mathbf{x}}, t) = E^i(\mathbf{x}, t) + \varepsilon^{ijk} \dot{x}_j B_k(\mathbf{x}, t) \quad (4)$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (5)$$

which actually corresponds to the Lorentz force and the first group of Maxwell equations. The result seems very strange since starting with a classical equation (the Newton's law), we end up with relativistic equations. In fact, "only" the first group of Maxwell equations is retrieved; the second group, according to Dyson, is a simple definition of matter. This is not a new idea; Le Bellac and Levy-Leblond [2] have already studied the Galilean invariance of these equations. Although with this approach, a Lagrangian or Hamiltonian structure is unnecessary, Hojman and Shepley [3] showed that by using a Helmholtz inverse variational problem under certain conditions, an action can be associated to these Feynman commutation relations.

The interpretation of the Feynman's derivation of Maxwell's equations has generated [3–12] a great interest among physicists. In particular, Tanimura [4] has generalized the Feynman's derivation in a Lorentz covariant form with a scalar time evolution parameter. An extension of the Tanimura's approach has been achieved [5] using the Hodge duality to derive the two groups of Maxwell's equations with a magnetic monopole in flat and curved spaces. In Ref. [6], the descriptions of relativistic and nonrelativistic particles in an electromagnetic field were studied, whereas in Ref. [7], a dynamical equation for spinning particles was proposed. A rigorous mathematical interpretation of Feynman's derivation associated with the inverse problem for Poisson dynamics has been formulated in Ref. [8]. Other works [10–12] have provided new insights into the Feynman's derivation of the Maxwell's equations. Recently [13], some of the authors embedded Feynman's derivation of the Maxwell's equation in the framework of noncommutative geometry. As Feynman's brackets can be interpreted as a deformation of Poisson brackets, we then showed that the Feynman brackets can be viewed as a generalization of the Moyal brackets defined over the tangent bundle space. We must also quote a new and very interesting study in noncommutative space by Carinena and Figueroa [14].

The noncommutation of the velocities in the presence of an electromagnetic field implies that the angular algebra symmetry, *e.g.* the $\mathfrak{so}(3)$ symmetry in the Euclidean space, is broken. If we restore such a symmetry, we point out the necessity of adding a Poincaré momentum \mathbf{M} to the simple angular momentum \mathbf{L} . Then, the direct consequence of this restoration is the generation of a Dirac magnetic monopole. The extension of these ideas to the covariant case in a Minkowski space is proposed in Section 2, where we show that this symmetry induces a magnetic angular momentum [15] as well as a new electric angular momentum. In Section 3, we include our work [16] in the natural generalization of quantum mechanics involving noncommutative space-time coordinates. This generalization was originally introduced by Snyder [17] as a short-distance regularization to improve the problem of infinite self-energies inherent in the quantum field theory. Due to the advent of the renormalization theory, this idea was not very popular until Connes [18] analyzed the Yang Mills theories on noncommutative space. Recently, a correspondence between a noncommutative gauge theory and a conventional gauge theory was introduced by Seiberg and Witten [19]. Noncommutative gauge theories were also found as being naturally related to the string theory and M-theory [20]. Applications of noncommutative theories were also found in condensed matter physics, for instance, in the quantum Hall effect [21]

and in the noncommutative Landau problem [22,23]. Then, the name of noncommutative quantum mechanics began to appear, notably with the works of [23,24]. In the context of noncommutative quantum mechanics, we present a summary of the paper [16] where the restoration of the $sO(3)$ symmetry leads to the introduction of a dual Dirac monopole in momentum space. This monopole was recently experimentally found in solid state physics [25]. Finally, from the study of the Dirac equation with an unspecified potential in an adiabatic approximation, we investigate [26] the link that exists between this formalism and the presence of a Berry phase, which plays an important role in the definition of the position operator.

2. Lorentz symmetry with Feynman brackets

One of the most important symmetry in physics is the spherical symmetry corresponding to the isotropy of physical space. This symmetry is related to the $sO(3)$ algebra. We showed within the Feynman brackets formalism that this symmetry is broken in the presence of an electromagnetic field. The restoration of this symmetry results in a Poincaré momentum and a Dirac monopole [15]. The generalization to the Lorentz symmetry is direct by assuming a particle of mass m moving in Minkowski space with position $x^\mu(\tau)$ ($\mu = 1, 2, 3, 4$) depending on parameter τ with the following commutation relations :

$$[x^\mu, \dot{x}^\nu] = \frac{\eta^{\mu\nu}}{m}, \quad (6)$$

where $\eta^{\mu\nu}$ is the metric. Following the same steps as those described in [27], we consider the angular momentum $L^{\mu\nu} = m(x^\mu \dot{x}^\nu - x^\nu \dot{x}^\mu)$. With the Feynman brackets, we recover the standard Lorentz Lie algebra and the transformation laws of position and velocity.

We then generalize the Feynman's approach by considering the following brackets [28] :

$$[\dot{x}^\mu, \dot{x}^\nu] = \frac{1}{m^2}(qF^{\nu\mu} + g^*F^{\nu\mu}), \quad (7)$$

where g is the magnetic charge of the magnetic monopole and the $*$ -operation is the Hodge duality. The derivative with respect to the time parameter of Eq. 7 leads to the following equation of motion

$$m\ddot{x}^\mu = qF^{\mu\nu}(\mathbf{x})\dot{x}_\nu + g^*F^{\mu\nu}(\mathbf{x})\dot{x}_\nu + G^\mu(\mathbf{x}). \quad (8)$$

The G field (which has no physical interpretation until now) also satisfies the equation $\partial^\mu G^\nu - \partial^\nu G^\mu = 0$. In the presence of the gauge fields, the Lorentz Lie algebra structure becomes more complicated.

$$\left\{ \begin{array}{l} [x^\mu, L^{\rho\sigma}] = \eta^{\mu\sigma}x^\rho - \eta^{\mu\rho}x^\sigma \\ [\dot{x}^\mu, L^{\rho\sigma}] = \eta^{\mu\sigma}\dot{x}^\rho - \eta^{\mu\rho}\dot{x}^\sigma + \frac{q}{m}(F^{\mu\sigma}\dot{x}^\rho - F^{\mu\rho}\dot{x}^\sigma) + \frac{g}{m}(*F^{\mu\sigma}\dot{x}^\rho - *F^{\mu\rho}\dot{x}^\sigma) \\ [L^{\mu\nu}, L^{\rho\sigma}] = \eta^{\mu\rho}L^{\nu\sigma} - \eta^{\nu\rho}L^{\mu\sigma} + \eta^{\mu\sigma}L^{\rho\nu} - \eta^{\nu\sigma}L^{\rho\mu} \\ \quad + q(x^\mu x^\rho F^{\nu\sigma} - x^\nu x^\rho F^{\mu\sigma} + x^\mu x^\sigma F^{\rho\nu} - x^\nu x^\sigma F^{\rho\mu}) \\ \quad + g(*F^{\nu\sigma}x^\mu x^\rho - *F^{\mu\sigma}x^\nu x^\rho + *F^{\rho\nu}x^\mu x^\sigma - *F^{\rho\mu}x^\nu x^\sigma) \end{array} \right. \quad (9)$$

Therefore, we introduce a generalized electromagnetic angular momentum $\mathcal{L}^{\mu\nu} = L^{\mu\nu} + M^{\mu\nu}$ in order to recover the usual algebra in the absence of electromagnetic fields. The requirement that the generalized electromagnetic angular momentum satisfies the usual algebra imposes constraints on the tensor M that can be solved leading to the following results

$$M^{ij} = q(F^{ij}x^k x_k - F^j_k x^k x^i - F_k^i x^k x^j) + g(*F^{ij}x^k x_k - *F^j_k x^k x^i - *F_k^i x^k x^j) \quad (10)$$

for the space components. The new angular momentum is, therefore, the sum of two contributions, a magnetic and an electric one :

$$\mathbf{M} = -q(\mathbf{x} \cdot \mathbf{B})\mathbf{x} + g(\mathbf{x} \cdot \mathbf{E})\mathbf{x} = \mathbf{M}_m + \mathbf{M}_e = -(\mathbf{x} \cdot \mathbf{P})\mathbf{x}, \quad (11)$$

where $\mathbf{M}_m = -q(\mathbf{x} \cdot \mathbf{B})\mathbf{x}$ and $\mathbf{M}_e = g(\mathbf{x} \cdot \mathbf{E})\mathbf{x}$ are the magnetic and electric angular momenta and $\mathbf{P} = q\mathbf{B} - g\mathbf{E}$.

Now, we require the Jacobi identity between the velocities, $[\dot{x}^\mu, [\dot{x}^\nu, \dot{x}^\rho]] + [\dot{x}^\nu, [\dot{x}^\rho, \dot{x}^\mu]] + [\dot{x}^\rho, [\dot{x}^\mu, \dot{x}^\nu]] = 0$ which yields the generalized Maxwell equations [28]

$$q(\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu}) + g(\partial^\mu *F^{\nu\rho} + \partial^\nu *F^{\rho\mu} + \partial^\rho *F^{\mu\nu}) = 0. \quad (12)$$

The projection of Eq. 12 on a three-dimensional space gives $q\nabla \cdot \mathbf{B} - g\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{P} = 0$, where \mathbf{P} can be considered either perpendicular to the vector \mathbf{x} or null. In both these cases, we have $\mathbf{M} = 0$. The Jacobi identity implies either there are no electric and magnetic monopoles or the two monopoles exactly compensate each other.

To break this duality symmetry, we no more require the Jacobi identity and introduce the tensor $N^{\mu\nu\rho}$ as

$$q(\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu}) + g(\partial^\mu *F^{\nu\rho} + \partial^\nu *F^{\rho\mu} + \partial^\rho *F^{\mu\nu}) = qgN^{\mu\nu\rho}, \quad (13)$$

which implies that $\nabla \cdot \mathbf{P} \neq 0$. The constraints on the Lie algebra can be fulfilled by a radial vector field centered at the origin solution: $\mathbf{P} = \frac{\mathbf{x}}{4\pi\|\mathbf{x}\|^3}$. As a consequence, we get a nonvanishing electromagnetic angular momentum

$$\mathbf{M} = \mathbf{M}_m + \mathbf{M}_e = -\frac{qg}{4\pi} \frac{\mathbf{x}}{\|\mathbf{x}\|}. \quad (14)$$

As the modulus of this momentum is radially constant, the magnetic and electric charges are not independent. It is the famous Dirac quantization condition connecting these two charges. It clearly appears that the vector $\mathbf{P} = q\mathbf{B} - g\mathbf{E}$ plays the same role as the magnetic field in the case without a dual field or in the three-dimensional theory.

Due to the fact that for these monopoles, the source of the fields is localized at the origin, we have

$$\nabla \cdot \mathbf{P} = [\dot{x}^i, [\dot{x}^j, \dot{x}^k]] + [\dot{x}^j, [\dot{x}^k, \dot{x}^i]] + [\dot{x}^k, [\dot{x}^i, \dot{x}^j]] \quad (15)$$

$$= q\nabla \cdot \mathbf{B} - g\nabla \cdot \mathbf{E} = \frac{qg}{4\pi} \left[x^l, \frac{x_l}{\|\mathbf{x}\|^3} \right] = qg\delta^3(\mathbf{x}). \quad (16)$$

For example, we can select $\mathbf{B} = \frac{g}{8\pi} \frac{\mathbf{x}}{\|\mathbf{x}\|}$ and $\mathbf{E} = -\frac{q}{8\pi} \frac{\mathbf{x}}{\|\mathbf{x}\|}$. We have found that \mathbf{M} is the new angular momentum, which is the sum of the Poincaré magnetic angular momentum [29] plus an electric angular momentum; \mathbf{B} being the field of a Dirac magnetic monopole [30] and \mathbf{E} the electric field of an electric Coulomb monopole.

In addition, we remark that the generalized angular moment $\mathcal{L} = m(\mathbf{x} \wedge \dot{\mathbf{x}}) - (\mathbf{x} \cdot \mathbf{P}) \cdot \mathbf{x}$ is conserved because the particle satisfies the usual equation of motion.

In conclusion of this section, it is interesting to note that with this same formalism, we can retrieve two groups of Maxwell equations and that this procedure of symmetry restoration has also been performed for Lorentz algebra in a curved space [13]. Another generalization of this approach can be found in a recent interesting work where the study of the Lorentz generators in N-dimensional Minkowski space has been proposed [31].

3. Noncommutative quantum mechanics

Let the momentum vector \mathbf{p} replace the velocity vector $\dot{\mathbf{x}}$ in the Feynman formalism presented before. Consider a quantum particle of mass m whose coordinates satisfy the deformed Heisenberg algebra

$$[x^i, x^j] = i\hbar q_\theta \theta^{ij}(\mathbf{x}, \mathbf{p}), \quad [x^i, p^j] = i\hbar \delta^{ij}, \quad [p^i, p^j] = 0,$$

where θ is a field that is *a priori* position- and momentum-dependent and q_θ is a charge characterizing the intensity of the interaction of the particle and the θ field. The commutation of the momentum implies that there is no external magnetic field. It is well known that these commutation relations can be obtained from the deformation of the Poisson algebra of classical observable with a provided Weyl-Wigner-Moyal product [32] expanded at the first order in θ .

3.1. Dual Dirac monopole in momentum space

The Jacobi identity $J(p^i, x^j, x^k) = 0$ implies an important property that the θ field is position-independent $\theta^{jk}(\mathbf{p})$. Then, one can consider the θ field as the dual of a magnetic field and q_θ as the dual of an electric charge. The fact that the field is homogeneous in space is an essential property for vacuum. In addition, one easily see that a particle in this field moves freely, *i.e.*, the vacuum field does not act on the motion of the particle in the absence of an external potential. The effect of the θ field is manifested only in the presence of a position-dependent potential. To look further at the properties of the θ field, consider the other Jacobi identity between the positions. We then have the equation of motion of the field

$$\frac{\partial \theta^{jk}(\mathbf{p})}{\partial p^i} + \frac{\partial \theta^{ki}(\mathbf{p})}{\partial p^j} + \frac{\partial \theta^{ij}(\mathbf{p})}{\partial p^k} = 0, \quad (17)$$

which is the dual equation of the Maxwell equation $\nabla \cdot \mathbf{B} = 0$. As we will see later, equation (17) is not satisfied in the presence of a monopole and this will have important consequences.

If we consider the position transformation

$$X^i = x^i + q_\theta a_\theta^i(\mathbf{x}, \mathbf{p}), \quad (18)$$

where a_θ is *a priori* position and momentum dependent, we are able to restore the usual canonical Heisenberg algebra

$$[X^i, X^j] = 0, \quad [X^i, p^j] = i\hbar \delta^{ij}, \quad [p^i, p^j] = 0. \quad (19)$$

The second commutation relation implies that a_θ is position-independent. The first commutation relation leads to the following expression of θ in terms of the dual gauge field a_θ :

$$\theta^{ij}(\mathbf{p}) = \frac{\partial a_\theta^i(\mathbf{p})}{\partial p^j} - \frac{\partial a_\theta^j(\mathbf{p})}{\partial p^i}, \quad (20)$$

which is dual of the standard electromagnetic relation in position space.

Consider now the problem of angular momentum. It is obvious that the angular momentum expressed according to the canonical coordinates satisfies the angular momentum algebra; however, it is not conserved,

$$\frac{d\mathcal{L}(\mathbf{X}, \mathbf{p})}{dt} = kq_\theta \mathcal{L} \wedge \Theta. \quad (21)$$

In the original (\mathbf{x}, \mathbf{p}) space, the usual angular momentum $L^i(\mathbf{x}, \mathbf{p}) = \varepsilon^i_{jk} x^j p^k$ does not satisfy this algebra. Therefore, it seems that there are no rotation generators in the (\mathbf{x}, \mathbf{p}) space. We will now prove that a genuine angular momentum can be defined only if θ is a nonconstant field.

Indeed, from the definition of angular momentum, we deduce the following commutation relations :

$$\begin{cases} [x^i, L^j] &= i\hbar\varepsilon^{ijk}x_k + i\hbar q\theta\varepsilon^j{}_{kl}p^l\theta^{ik}(\mathbf{p}), \\ [p^i, L^j] &= i\hbar\varepsilon^{ijk}p_k, \\ [L^i, L^j] &= i\hbar\varepsilon^i{}_{jk}L^k + i\hbar q\theta\varepsilon^i{}_{kl}\varepsilon^j{}_{mn}p^l p^n\theta^{km}(\mathbf{p}), \end{cases} \quad (22)$$

particularly showing that the $\mathfrak{so}(3)$ algebra is broken. To restore the angular momentum algebra, consider the transformation law

$$L^i \rightarrow \mathbb{L}^i = L^i + M_\theta^i(\mathbf{x}, \mathbf{p}); \quad (23)$$

then, the usual $\mathfrak{so}(3)$ algebra needs to be satisfied. Then, it can be shown that this condition can be fulfilled by a dual Dirac monopole defined in momentum space¹

$$\Theta(\mathbf{p}) = \frac{q\theta}{4\pi} \frac{\mathbf{p}}{\|\mathbf{p}\|^3}, \quad (24)$$

where we introduced the dual magnetic charge $q\theta$ associated with the Θ field. Consequently, we have $\mathbf{M}_\theta(\mathbf{p}) = -\frac{q\theta g\theta}{4\pi} \frac{\mathbf{p}}{\|\mathbf{p}\|}$, which is the dual of the famous Poincaré momentum introduced in position space. Then, the generalized angular momentum

$$\mathbb{L} = \mathbf{x} \wedge \mathbf{p} - \frac{q\theta g\theta}{4\pi} \frac{\mathbf{p}}{\|\mathbf{p}\|} \quad (25)$$

is a genuine angular momentum satisfying the usual algebra. One can check that it is a conserved quantity for a free particle.

The duality between the monopole in momentum space and the Dirac monopole is due to the symmetry of the commutation relations in noncommutative quantum mechanics, where $[x^i, x^j] = i\hbar q\theta\varepsilon^{ijk}\Theta_k(\mathbf{p})$, and the usual quantum mechanics in a magnetic field, where $[\hat{x}^i, \hat{x}^j] = i\hbar q\varepsilon^{ijk}B_k(\mathbf{x})$. Therefore, the two gauge fields $\Theta(\mathbf{p})$ and $B(\mathbf{x})$ are a dual of each other.

Note that in the presence of the dual monopole, the Jacobi identity (17) does not hold

$$\left[x^i, \left[x^j, x^k \right] \right] + \left[x^j, \left[x^k, x^i \right] \right] + \left[x^k, \left[x^i, x^j \right] \right] = -q\theta\hbar^2 \frac{\partial\Theta^i(\mathbf{p})}{\partial p_i} = -4\pi q\theta\hbar^2 g\theta\delta^3(\mathbf{p}). \quad (26)$$

This term is responsible for the violation of the associativity, which is only restored if the following quantification equation is satisfied : $\int d^3p \frac{\partial\Theta^i}{\partial p_i} = \frac{2\pi n\hbar}{q\theta}$, leading to $q\theta g\theta = \frac{n\hbar}{2}$, which is in complete analogy with Dirac's quantization. We should also note that a new insight into fuzzy monopoles in the context of a nonconstant noncommutativity in 2D field theories has very recently been developed [35].

3.2. Link with the Berry phase

In quantum mechanics, this construction may look formal because it is always possible to introduce commuting coordinates with the help of the transformation $\mathbf{R} = \mathbf{x} - \mathbf{p} \wedge \mathbf{S}/p^2$. The angular momentum is then $\mathbf{J} = \mathbf{R} \wedge \mathbf{p} + \mathbf{S}$ and it satisfies the usual algebra, whereas the potential energy term in the Hamiltonian becomes $V(\mathbf{R} + \mathbf{p} \wedge \mathbf{S}/p^2)$, which contains spin-orbit interactions. In fact, the inverse procedure is usually more efficient. If we consider a Hamiltonian with a particular spin-orbit interaction, a trivial Hamiltonian with a nontrivial dynamics due to the noncommutative coordinates algebra can be obtained. This procedure has

¹ This result has been already found in an other context [33, 34].

been applied with success to the study of adiabatic transport in semiconductors with spin-orbit couplings [25]. The difficulty is now to decide which one of the two position operators \mathbf{x} or \mathbf{R} gives rise to the real mean trajectory of the particle. In fact, it is well known that \mathbf{R} does not have the correct properties of a position operator for a relativistic particle. As we shall see, this point is very important when considering the nonrelativistic limit as we predict an effect similar to the Thomas precession but with regard to the velocity.

It should also be noted that other recent theoretical works concerning the anomalous Hall effect in two-dimensional ferromagnets predict a topological singularity in the Brillouin zone [36]. In addition, a monopole in the crystal momentum space was experimentally discovered and interpreted in terms of an Abelian Berry curvature [25].

3.2.1. Dirac equation The Dirac's Hamiltonian for a relativistic particle of mass m has the form $\hat{H} = \alpha \cdot \mathbf{p} + \beta m + \hat{V}(\mathbf{R})$, where \hat{V} is an operator that acts only on the orbital degrees of freedom. Using the Foldy-Wouthuysen unitary transformation, we get the following transformed Hamiltonian

$$U(\mathbf{p})\hat{H}U(\mathbf{p})^+ = E_p\beta + U(\mathbf{p})\hat{V}(i\hbar\partial_{\mathbf{p}})U(\mathbf{p})^+. \quad (27)$$

The kinetic energy is now diagonal, whereas the potential term becomes $\hat{V}(\mathbf{D})$ with the covariant derivative defined by $\mathbf{D} = i\hbar\partial_{\mathbf{p}} + \mathbf{A}$ and with the gauge potential $\mathbf{A} = i\hbar U(\mathbf{p})\partial_{\mathbf{p}}U(\mathbf{p})^+$. We now consider the adiabatic approximation that neglects the interband transition. We then keep only the block diagonal matrix element in the gauge potential and project it on the subspace of positive energy. This projection cancels the zitterbewegung, which corresponds to an oscillatory movement around the mean position of the particle that combines the positive and negative energies. In this way, we obtain a nontrivial gauge connection allowing us to define a new position operator \mathbf{r} for this particle

$$\mathbf{r} = i\hbar\partial_{\mathbf{p}} + i\hbar\mathcal{P}(U\partial_{\mathbf{p}}U^+), \quad (28)$$

where \mathcal{P} is a projector on the positive energy subspace. It is straightforward to prove that the anomalous part of the position operator can be interpreted as a Berry connection in momentum space. In this context, the θ field we postulated in [16] appears as a consequence of the adiabatic motion of a Dirac particle and corresponds to a non-Abelian gauge curvature satisfying the relation

$$\theta^{ij}(\mathbf{p}, \sigma) = \partial_{p^i}A^j - \partial_{p^j}A^i + [A^i, A^j]. \quad (29)$$

The commutation relations between the coordinates are then expressed as

$$[x^i, x^j] = i\hbar\theta^{ij}(\mathbf{p}, \sigma), \quad (30)$$

which has very important consequences as it implies the nonlocalizability of the spinning particles. This is an intrinsic property and is not related to the creation of a pair during the measurement process (for a detailed discussion of this very important point, see [37]).

Note that it is possible to generalize the construction of the position operator for a particle with unspecified $n/2$ ($n > 1$) spin through the Bargmann-Wigner equations. In this way, the general position operator \mathbf{r} for spinning particles is

$$\mathbf{r} = i\hbar\partial_{\mathbf{p}} + \frac{c^2(\mathbf{p} \wedge \mathbf{S})}{E_p(E_p + mc^2)}. \quad (31)$$

The generalization of (30) is then

$$[x^i, x^j] = i\hbar\theta^{ij}(\mathbf{p}, \mathbf{S}) = -i\hbar\varepsilon_{ijk}\frac{c^4}{E_p^3}\left(mS^k + \frac{p^k(\mathbf{p}\cdot\mathbf{S})}{E_p + mc^2}\right). \quad (32)$$

For a massless particle, we recover the relation $\mathbf{r} = i\hbar\partial_{\mathbf{p}} + \mathbf{p} \wedge \mathbf{S}/p^2$ with the commutation relation giving rise to the monopole $[x^i, x^j] = i\hbar\theta^{ij}(\mathbf{p}) = -i\hbar\varepsilon_{ijk}\lambda\frac{p^k}{p^3}$. The momentum space monopole we introduced in [16] in order to construct a genuine angular momenta has a very simple physical interpretation. It corresponds to the Berry curvature resulting from an adiabatic process of massless particle with helicity λ . It is not surprising that a massless particle has a monopole Berry curvature as it is well known that the band-touching point acts as a monopole in momentum space [38]. This is precisely the case for massless particles for which the positive and negative energy bands are degenerate in $p = 0$. The monopole appears as a limiting case of a more general non-Abelian Berry curvature arising from an adiabatic process of massive spinning particles. For $\lambda = \pm 1$, we have the position operator of the photon, whose noncommutativity property agrees with the weak localizability of the photon, which is certainly an experimental fact.

The spin-orbit coupling term in (31) is a very small correction to the usual operator in the particle physics context, but it may be strongly enhanced and observed in solid state physics because the spin-orbit effect is much more important than in the vacuum. For instance, in narrow-gap semiconductors, the equations of the bands theory are similar to the Dirac equation with the forbidden gap E_G between the valence and conduction bands instead of the Dirac gap $2mc^2$ [39]. The monopole in momentum space predicted and observed in semiconductors results from the limit of the vanishing gap $E_G \rightarrow 0$ between the valence and conduction bands.

It is also interesting to look at the symmetry properties of the position operator with respect to the group of spatial rotations. In terms of commutative coordinates \mathbf{R} , the angular momentum is by definition $\mathbf{J} = \mathbf{R} \wedge \mathbf{p} + \mathbf{S}$, whereas in terms of the noncommutative coordinates, the angular momentum reads $\mathbf{J} = \mathbf{r} \wedge \mathbf{p} + \mathbf{M}$, where

$$\mathbf{M} = \mathbf{S} - \mathbf{A} \wedge \mathbf{p}. \quad (33)$$

One can explicitly check that in terms of the noncommutative coordinates, the relation $[x^i, J^j] = i\hbar\varepsilon^{ijk}x_k$ is satisfied; therefore \mathbf{r} , like \mathbf{R} , transforms as a vector under space rotations, but $d\mathbf{R}/dt = c\alpha$ is physically unacceptable. For a massless particle, Eq. 33 leads to the Poincaré momentum associated with the monopole in momentum space deduced in [16].

3.2.2. Physical applications We are interested to look at some physical properties of the noncommuting position operator. Let us consider the equation of motion of a particle in an arbitrary potential. Due to the Berry phase in the definition of position, the equation of motion will change. In order to compute a commutator like $[x^k, V(x)]$, one can consider the semiclassical approximation $[x^k, V(x)] = i\hbar\partial_l V(x)\theta^{kl} + O(\hbar^2)$, which gives the following equations of motions

$$\dot{\mathbf{r}} = \frac{\mathbf{p}}{E_p} - \dot{\mathbf{p}} \wedge \theta, \quad \text{and} \quad \dot{\mathbf{p}} = -\nabla V(\mathbf{r}) \quad (34)$$

with $\theta^i = \varepsilon^{ijk}\theta_{jk}/2$. These equations are the relativistic generalization of the equations found in [25] that leads to the spin Hall effect in the context of semiconductors. Equations 34 have also the same form as those found in [40] in the context of condensed matter physics for a particle (without spin) propagating in a periodic potential. The difference in this case relies on the fact that the two Berry phases have a completely different physical origin as that in [40]; the Berry phase is only due to the periodic potential.

An important physical application of our theory concerns the nonrelativistic limit of a charged spinning Dirac particle in a potential $\hat{V}(\mathbf{r})$. In this limit, we obtain

$$\tilde{H}(\mathbf{R}, \mathbf{p}) \approx mc^2 + \frac{p^2}{2m} + \hat{V}(\mathbf{R}) + \frac{e\hbar}{4m^2c^2}\sigma \cdot (\nabla\hat{V}(\mathbf{r}) \wedge \mathbf{p}), \quad (35)$$

which is a Pauli Hamiltonian with a spin-orbit interaction term. As shown in [41], the Born-Oppenheimer approximation of the Dirac equation leads to the same nonrelativistic Hamiltonian as a consequence of the nonrelativistic Berry phase $\theta^{ij} = -\varepsilon_{ijk}\sigma^k/2mc^2$. Note that in [41], it was also proved that the adiabaticity condition (which neglects the non-diagonal matrix elements of \hat{V}) is satisfied for slowly varying potentials as long as $L \gg \lambda$, where L is the length scale over which $\hat{V}(\mathbf{r})$ varies and λ is the de Broglie wavelength of the particle. As a consequence of Eq. 35, we deduce the velocity associated with the usual Galilean-Schrödinger position operator \mathbf{R}

$$\frac{dX^i}{dt} = \frac{p^i}{m} + \frac{e\hbar}{4m^2c^2}\varepsilon^{ijk}\sigma_j\partial_k\hat{V}(\mathbf{r}), \quad (36)$$

whereas the nonrelativistic limit of Eq. 34 leads to the following velocity operator

$$\frac{dx^i}{dt} = \frac{p^i}{m} + \frac{e\hbar}{2m^2c^2}\varepsilon^{ijk}\sigma_j\partial_k\hat{V}(\mathbf{r}); \quad (37)$$

this result predicts an enhancement of the spin-orbit coupling when the new position operator is considered. One can appreciate the similarity between our result and the Thomas precession. Indeed, this result offers another manifestation beside the Thomas precession of the difference between the Galilean limit (leading to Eq. 36) and the nonrelativistic limit (leading to Eq. 37).

The ultrarelativistic limit gives us another example of topological spin transport. Experimentally, a topological spin transport has already been observed in the case of the photon propagation in an inhomogeneous medium [42], where the right and left circular polarization propagate along different trajectories in a waveguide (the transverse shift is observable due to the multiple reflections), a phenomenon interpreted quantum mechanically as arising from the interaction between the orbital momentum and the spin of the photon [42]. To interpret the experiments, these authors introduced a complicated phenomenological Hamiltonian. Our approach provides a new satisfactory interpretation as this effect, also called the optical Magnus effect, is now explained in terms of the noncommutative property of the position operator that contains the spin-orbit interaction. In this sense, this effect is just the ultrarelativistic spin Hall effect. Note that the adiabaticity criteria has been proved to be valid in [43]. To illustrate our purpose, consider the simple photon Hamiltonian in the inhomogeneous medium $H = pc/n(r)$. The equations of motion $\dot{x} = \frac{1}{i\hbar}[x, H]$ and $\dot{p} = \frac{1}{i\hbar}[p, H]$ in the semiclassical approximation leads to following relation between velocity and momentum

$$\frac{dx^i}{dt} = \frac{c}{n} \left(\frac{p^i}{p} + \frac{\lambda\varepsilon^{ijk}p_k}{p^2} \frac{\partial \ln n}{\partial x^j} \right), \quad (38)$$

which contains an unusual contribution due to the Berry phase. As a consequence, the velocity is no more equal to c/n . Equations 38 are the same as those introduced phenomenologically in [42], but here, they are rigorously deduced from different physical consideration. Similar equations are also given in [44] where the optical Magnus effect is also interpreted in terms of a monopole Berry curvature, but in the context of geometrical optics. Our theory is generalizable to the photon propagation in a nonisotropic medium; a situation that is mentioned in [42], but could not be studied with their phenomenological approach.

4. Conclusion

In this communication, we have tried to stress the close link, between the Feynman's approach of the Maxwell equations and noncommutative quantum mechanics. In particular, we show that the restoration of broken symmetries leads to the appearance of Dirac monopoles either in configuration space or momentum space. Actually, the generalization of Feynman's ideas to the

case of the noncommutative quantum mechanics is interesting because it naturally leads to the promotion of the θ parameter to a $\theta(\mathbf{p})$ field. As a physical realization of the noncommutative theory, we showed that the $\theta(\mathbf{p})$ field can be interpreted in some circumstances as a Berry curvature associated with a Berry phase expressed in momentum space. This was shown in the context of Dirac particle and photon propagation. Particular important is the fact that the Berry phase leads to a new physically interesting effect called spin Hall effect.

Recently, noncommutative quantum mechanics has been the topic of several other works in particle and condensed matter physics. For instance, it was shown that the Berry phase (which is a spin-orbit interaction) of the photon influences the geometric optics equations leading to the Magnus effect of light [26]. Actually, the spin-orbit contribution on the propagation of light has led to a generalization of geometric optics called geometric spinoptics [45]. Other applications concern the propagation of Dirac particles and photons in a static gravitational field [46,47] and the semiclassical equations of motion of electrons in a solid [40,48,49]. In semiconductor physics, it has also been found that a noncommutative geometry underlies the semiclassical dynamics of electrons in an external electric field. Here, the noncommutativity property is again the consequence of a Berry phase inducing a purely topological and dissipationless spin current (intrinsic spin Hall effect) [25]. More generally, we mention the current efforts carried out in order to better understand the close relation existing between the noncommutative geometry and the geometric phase [50].

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