A numerical algorithm to find the equilibrium of a conservative system

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Why this algorithm ?

- **Aim:** determine the exact location of a dynamical equilibrium (forced oscillations only) corresponding to a resonance
- **Problem:** - accurate (realistic) simulations give some parasitical (free) librations supposed to be damped
  - the initial condtions given by analytical studies are not accurate enough
- **Solution:** this algorithm use NAFF[4] (Numerical Analysis of the Fundamental Frequencies) for the identification of the free and forced oscillations, the former being iteratively removed from the solution by carefully choosing the initial conditions.

History

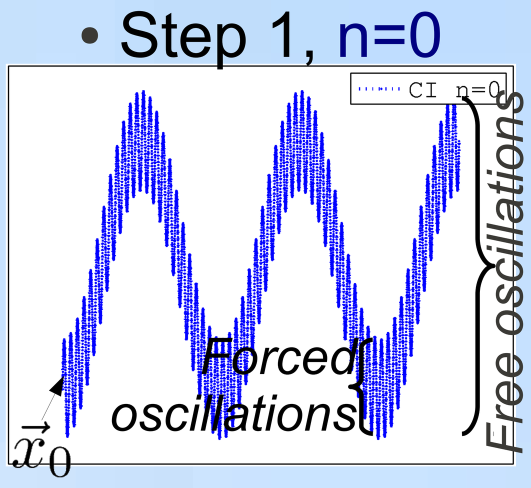
This algorithm has been used in some works yet:

- Rotation problem [3,5,7]
- Exoplanetary dynamics [1]
- Dynamics of a probe around Vesta [2]
- ...

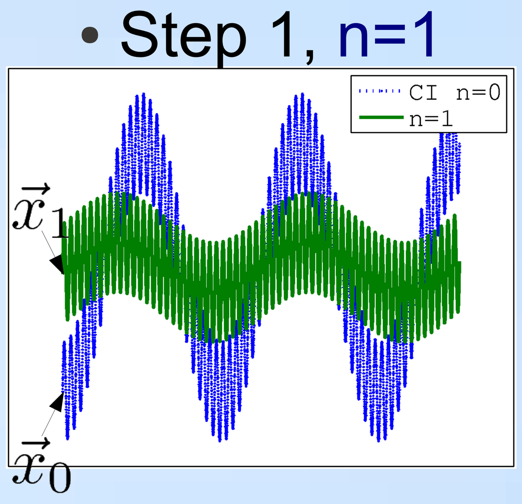
Here[6] we provide the convergence proof and give the quadratic convergence in the Hamiltonian case.

The Algorithm

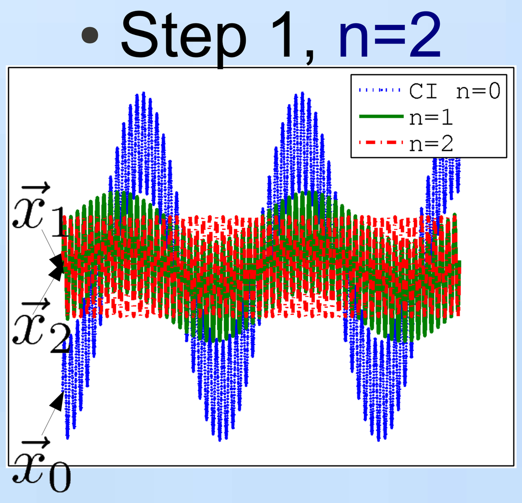
- Step 1, n=0



- Step 1, n=1



- Step 1, n=2



Step 0 Initialisation	• Choose an initial condition $\vec{x}_0$ close to the equilibrium (e.g. using an analytical solution) and set n=0
Step 1 Integration	• Obtain the evolution of $\vec{x}$ with respect to the time: $\vec{x}(t)$
Step 2 Identification	• Express the variable $\vec{x}$ of the problem by a quasi-periodic decomposition (e.g. using NAFF[4]) • Isolate the free (to remove) and the forced (to keep) oscillations
Step 3 New Initial Conditions	• Remove the free oscillations from the initial conditions $\vec{x}_n$ to obtain the new initial conditions $\vec{x}_{n+1}$ $\vec{x}_{n+1} = \vec{x}_n - \sum_{i \in \text{Free terms}} \text{Ampl}_i \cos(\text{Phase}_i)$ • Increase n: n=n+1 • Go to Step 1 until $\lVert \vec{x}_{n+1} - \vec{x}_n \rVert < \varepsilon_{IC}$ or until $\text{Ampl}_i < \varepsilon_{\text{Ampl}} \; \forall i$

Convergence proof (for Hamiltonian case)[6]

Let $\dot{\vec{X}} = f(\vec{X}) + g(\vec{X}, t)$ with the solution

$$\phi(t;\vec{X}) = \sum_{m \in \mathbb{Z}} \phi_{0m}(\vec{X}) e^{\imath \nu m t} + \sum_{l \neq 0, m \in \mathbb{Z}} \phi_{lm}(\vec{X}) e^{\imath(\omega l + \nu m)t} := S(t;\vec{X}) + L(t;\vec{X})$$

and the fixed point $\vec{X}_\infty$ such as $\phi(t;\vec{X}_\infty) = S(t;\vec{X}_\infty)$.

Assuming an Hamiltonian framework and an initial condition $\vec{X}_0$ such as $|\vec{X}_0 - \vec{X}_\infty| < 1$.

Then, the algorithm gives a sequence $(\vec{X}_n)_n$ where $\vec{X}_n \underset{n \to \infty}{\longrightarrow} \vec{X}_\infty$ and the convergence rate is quadratic $|\vec{X}_{n+1} - \vec{X}_\infty| \propto |\vec{X}_n - \vec{X}_\infty|^2$

Idea of proof (one dim. to simplify)

We have to prove that x_∞ is an attractor:

$$|f'(x_\infty)| < 1 \Longleftrightarrow \lim_{x \to x_\infty} \frac{\partial_x S(0;x)/\partial_x L(0;f(x))}{\partial_x S(0;f(x))/\partial_x L(0;f(x)) + 1} = 0$$

Using the d'Alembert rule we can state that (x close to x_∞)

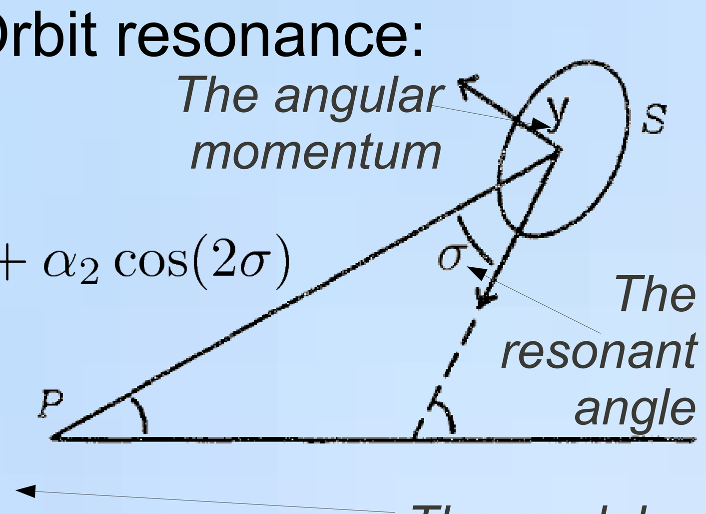
$$S(0;x) \sim x_\infty + a|x - x_\infty| + \dots \quad \text{and} \quad L(0;x) \sim x_\infty + b\sqrt{|x - x_\infty|} + \dots$$

Then, $\partial_x S(0;x)/\partial_x L(0;f(x)) \underset{x \to x_\infty}{\longrightarrow} 0$ and the convergence rate is quadratic.

An example

Earth-Moon Spin-Orbit resonance:

$$H(y,\sigma,L,\lambda) = \frac{y^2}{2} - \epsilon\Big[\alpha_1 \cos(2\sigma + \lambda) + \alpha_2 \cos(2\sigma) + \alpha_3 \cos(2\sigma - \lambda) + \alpha_4 \cos(2\sigma - 2\lambda) + \alpha_5 \cos(2\sigma - 3\lambda)\Big] + L - y$$



- Step 0: the averaged Hamiltonian

$$\bar{H}(y,\sigma,L,-) = \frac{(y-1)^2}{2} - \epsilon\alpha_2 \cos(2\sigma) + L$$

gives the first initial condition: $y_0 = 0 \,, \quad \sigma_0 = 0$

- Step 1 (numerical integration) + Step 2 (NAFF)

Frequency decomposition of y(t)

N	Ampl.	Freq.	Phase (rad)	Id.
1	1.000 000	0.000 00	2.462×10^{-14}	
2	$7.536\,994 \times 10^{-4}$	1.000 00	3.141 588	λ
3	$8.010\,500 \times 10^{-5}$	$2.616\,86 \times 10^{-2}$	0.000 008	u
4	$4.392\,058 \times 10^{-6}$	2.000 00	3.141 587	2λ
5	$3.328\,440 \times 10^{-7}$	3.000 00	3.141 587	3λ
...	...	...	...	...

- Step 3: new initial conditions

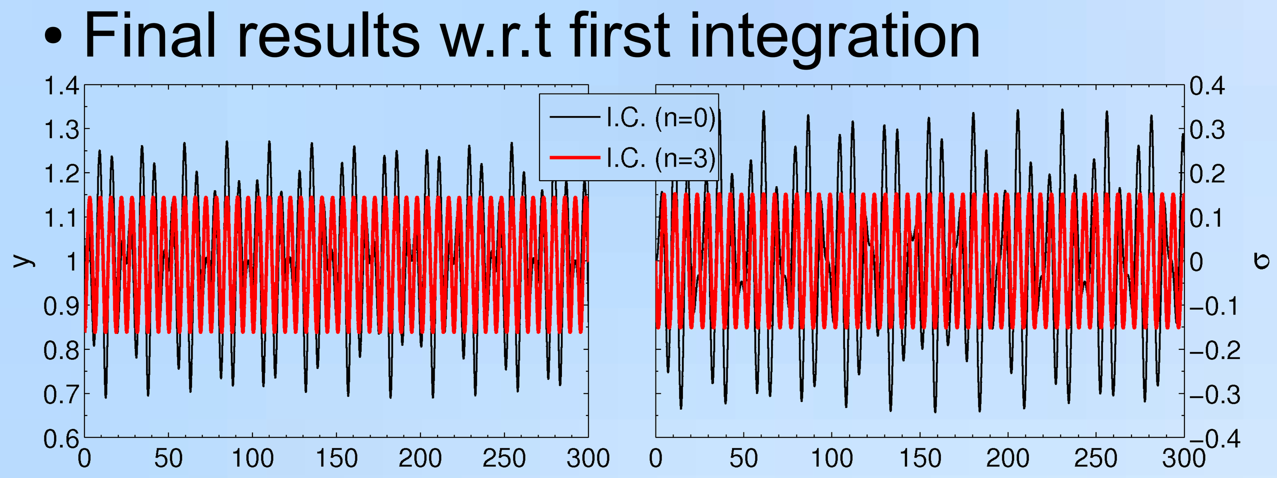
$$y_1 = y_0 - 8.010500 10^{-5} \cos(0.000008) = 0.999920$$

$$\sigma_1 = \sigma_0 - 3.061107 10^{-3} \cos(1.570796) = 3.106991 10^{-10}$$

- Some iterations of the algorithm

n		I.C.	Ampl. - free term	$\omega^\bullet$
0	y	1.000 000 000	$8.010\,500 \times 10^{-5}$	$2.616\,864 \times 10^{-2}$
	σ	0.000 000 000	$3.061\,106 \times 10^{-3}$	$2.616\,864 \times 10^{-2}$
1	y	0.999 919 905	$7.770\,876 \times 10^{-10}$	$2.616\,870 \times 10^{-2}$
	σ	$3.106\,991\,059 \times 10^{-10}$	$2.969\,532 \times 10^{-8}$	$2.616\,870 \times 10^{-2}$
2	y	0.999 919 904	$1.264\,196 \times 10^{-15}$	$2.616\,860 \times 10^{-2}$
	σ	$3.490\,283\,085 \times 10^{-15}$	$4.830\,976 \times 10^{-14}$	$2.616\,860 \times 10^{-2}$

- Final results w.r.t first integration



References :

[1] Couetdic J. et al (2010), Astronomy and Astrophysics, 519

[2] Delsate N. (2011), Planetary and Space Science, 59

[3] Dufey J. et al. (2009), Icarus, 203

[4] Laskar J. (1993), Celestial Mechanics and Dynamical Astronomy, 56

[5] Noyelles B. (2009), Icarus, 202

[6] Noyelles et al., arXiv:1101.2138

[7] Robutel P. et al. (2011), Icarus, 211